BIANCHI GROUPS AND AUTOMORPHISMS OF RANK-FOUR K3 SURFACES

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Abstract. We relate the arithmetic of Bianchi groups to automorphism groups of Picard-rank-four $K3$ surfaces. Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, and let $S_K = \mathrm{Herm}_2(\mathcal{O}_K)$ be the rank-four lattice of 2×2 Hermitian matrices over $\mathcal{O}_K$, equipped with the quadratic form $2\det$. For an odd integer $N \geq 1$, we consider a very general $S_K(2N)$-polarized $K3$ surface $X_{K,2N}$.

We prove that its automorphism group is commensurable with a level-$2N$ congruence subgroup of the Bianchi group. Furthermore, we also obtain exact realizations of congruence subgroups as full automorphism groups. Namely, if $K = \mathbb{Q}(i)$ or $K = \mathbb{Q}(\sqrt{-p})$, where p is prime, then

$$\mathrm{Aut}(X_{K,2}) \cong P\Gamma_K(2).$$

Thus, for every prime p, the projective principal congruence subgroup of level 2 over $\mathcal{O}_{\mathbb{Q}(\sqrt{-p})}$ occurs as the full automorphism group of a Picard-rank-four $K3$ surface. At higher levels, the full automorphism group may be either the projective principal congruence subgroup or the strictly larger projective level subgroup $\mathrm{Bi}_K(2N)$, depending on the arithmetic of the primes dividing the level.

We further explain these arithmetic groups geometrically. The surfaces $X_{K,2}$ arise as deformations of the Kummer surfaces $\mathrm{Km}(E_K \times E_K)$, yielding explicit double-cover models and genus-one fibrations. For $K = \mathbb{Q}(\sqrt{-2})$ and $K = \mathbb{Q}(\sqrt{-7})$, the automorphism group is generated by Mordell–Weil translations associated with genus-one fibrations coming from cusps, together with the covering involution. For $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-3})$, we construct complete-intersection models in products of projective spaces and show that their automorphism groups are generated by deck involutions.

Contents

1. Introduction

A fundamental question in algebraic geometry, broadly posed by Mazur [10, Section 7], asks to what extent the automorphism groups of projective varieties are governed by arithmetic groups. However, Totaro demonstrated that this expectation fails, even for $K3$ surfaces of Picard rank four: there exist Picard-rank-four $K3$ surfaces whose automorphism groups are not commensurable with any arithmetic group [15, Theorem 6.1 and Example 6.3]. Despite this general failure, there are also positive realization results. For

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example, the first author and Lee [5] have shown how to realize congruence subgroups of the modular group as the automorphism group of certain $K3$ surfaces. These explicit constructions prove that while arithmeticity is not guaranteed, highly structured arithmetic groups can still be captured by the geometry of specific $K3$ surfaces.

The purpose of this paper is to push this positive realization program into the realm of three-dimensional hyperbolic geometry. The example constructed by the first author and Lee has Picard rank three, so that the projectivized positive cone of the Néron–Severi lattice is a model of the hyperbolic plane $\mathbb{H}^2$, and the automorphism group is realized through its action on this space. In Picard rank four, the corresponding projectivized positive cone is instead a model of hyperbolic three-space $\mathbb{H}^3$, providing a natural source of discrete groups of hyperbolic isometries. For any $K3$ surface, the action on the Néron–Severi lattice gives a representation $\mathrm{Aut}(X) \longrightarrow O(\mathrm{NS}(X))$. In the cases considered in this paper, this representation is injective by Lemma 2.6. Hence we can identify $\mathrm{Aut}(X)$ with subgroup of $O(\mathrm{NS}(X))$. We construct explicit examples of Picard-rank-four $K3$ surfaces whose automorphism groups are not merely abstract discrete groups of hyperbolic isometries, but are described precisely as congruence subgroups of Bianchi groups; see Theorem 1.

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. We consider the lattice

$$S_K := \mathrm{Herm}_2(\mathcal{O}_K) = \left\{ A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} : a, c \in \mathbb{Z},\ b \in \mathcal{O}_K \right\}, \qquad q(A) := 2\det(A).$$

This lattice has signature $(1,3)$, and the Bianchi group $\mathrm{Bi}_K := \mathrm{PGL}_2(\mathcal{O}_K)$ acts on S_K by Hermitian conjugation; see Subsection 3.1 for this identification. For every odd integer $N \geq 1$, we construct very general $K3$ surfaces $X_{K,2N}$ satisfying $\mathrm{NS}(X_{K,2N}) \cong S_K(2N)$.

The very generality assumption, together with Lemma 2.4 and the global Torelli theorem, gives a concrete criterion for which Bianchi isometries are realized by automorphisms of $X_{K,2N}$. Namely, Proposition 3.8 shows that an element of Bi_K can be induced by an automorphism only if its action on the discriminant group $A_{S_K(2N)}$ is multiplication by a single global sign ± 1. This condition is arithmetic and can be expressed in terms of congruence subgroups.

For $m \geq 1$, we define the projective level-m congruence subgroup

$$\mathrm{Bi}_K(m) := \ker(\mathrm{PGL}_2(\mathcal{O}_K) \longrightarrow \mathrm{PGL}_2(\mathcal{O}_K/m\mathcal{O}_K)).$$

We also consider the finer projective principal congruence subgroup

$$P\Gamma_K(m) := \left\{ [M] \in \mathrm{PGL}_2(\mathcal{O}_K) \;\middle|\; \begin{array}{c} \text{some representative } M \in \mathrm{GL}_2(\mathcal{O}_K) \\ \text{satisfies } M \equiv I_2 \pmod{m\mathcal{O}_K} \end{array} \right\}.$$

There is always an inclusion $P\Gamma_K(m) \subseteq \mathrm{Bi}_K(m)$. In particular, when $m = 2$ and $\mathcal{O}_K^\times = \{\pm 1\}$, the two groups agree if 2 is split or inert, whereas they may differ if 2 is ramified; see Proposition 3.7.

Our first result compares the automorphism group with the arithmetic congruence subgroups introduced above. It shows that the part of the automorphism group lying in the Bianchi group is contained in $\mathrm{Bi}_K(2N)$ and, under an additional unramifiedness assumption at 2, has explicitly bounded index. The index bound measures the discrepancy between the congruence subgroup determined by the level structure and the subgroup satisfying the global-sign condition on the discriminant group from Proposition 3.8. This is a uniform estimate that depends only on level $2N$.

Theorem 1 (Corollaries 3.9, 3.15)**.** *Let K be an imaginary quadratic field with fundamental discriminant $-D$, and let $N \geq 1$ be an odd integer such that every prime divisor of N is unramified in K. Let $X_{K,2N}$ be a very general $S_K(2N)$-polarized $K3$ surface. Then the following statements hold:*

(1) $\mathrm{Bi}_K \cap \mathrm{Aut}(X_{K,2N}) \subseteq \mathrm{Bi}_K(2N)$.

(2) *If 2 is also unramified in K, then*

$$[\mathrm{Bi}_K(2N) : \mathrm{Bi}_K \cap \mathrm{Aut}(X_{K,2N})] \leq 2^{\omega(N)},$$

where ω denotes the number of distinct prime divisors of N.

The intersections in parts (1) and (2) are necessary because, with our definitions, $\mathrm{Aut}(X_{K,2N})$ is not a priori realized as a subgroup of the Bianchi group; see Proposition 3.3. Thus, the theorem first identifies the portion of the automorphism group lying in Bi_K and then compares it with the projective level subgroup $\mathrm{Bi}_K(2N)$.

Our second result determines the full automorphism group at level 2 for the fields $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-p})$, where p is prime.

Theorem 2. *Let $K = \mathbb{Q}(i)$ or $K = \mathbb{Q}(\sqrt{-p})$, where p is a prime. Then* $\mathrm{Aut}(X_{K,2}) \cong P\Gamma_K(2)$.

The proof depends on the arithmetic behavior of the prime 2 in K. When 2 is unramified, the result follows from the local discriminant-form computation and the index estimate in Theorem 1. When 2 is ramified, a separate level-two analysis is required; this is carried out in Lemmas 3.18–3.20. One must also account for the fact that the natural lattice representation places the automorphism group inside the extended Bianchi group a priori. Proposition 3.3 and Lemma 3.20 show that the additional extended-Bianchi cosets do not produce further automorphisms in the cases under consideration.

As a consequence of Theorem 2, for every prime p, the group $P\Gamma_{\mathbb{Q}(\sqrt{-p})}(2)$ occurs as the full automorphism group of a Picard-rank-four $K3$ surface.

We next give exact realizations at higher level. Here two distinct phenomena occur. In the first family, the full automorphism group is the projective principal congruence subgroup. In the second, it is the strictly larger projective level subgroup $\mathrm{Bi}_K(2N)$.

Theorem 3 (Theorems 3.21, 3.22)**.** *The following statements hold under the normalized Bianchi action:*

(1) *Let p be a prime satisfying $p \equiv 7 \pmod 8$, and let $K = \mathbb{Q}(\sqrt{-p})$. Suppose that $N = q^e$, where $q \equiv 3 \pmod 4$ is an odd prime that splits in K. Then*

$$\mathrm{Aut}(X_{K,2N}) \cong P\Gamma_K(2N).$$

(2) *Let $K = \mathbb{Q}(\sqrt{-p})$, where $p > 3$ is a prime satisfying $p \equiv 3 \pmod 4$. Suppose that $N > 1$ is odd and that every prime divisor $\ell \mid N$ is inert in K and satisfies $\ell \equiv 1 \pmod 4$. Then*

$$P\Gamma_K(2N) \subsetneq \mathrm{Bi}_K(2N) \cong \mathrm{Aut}(X_{K,2N}).$$

The two parts of Theorem 3 exhibit a genuine higher-level dichotomy. In part (1), where the prime dividing N splits in K, the full automorphism group is the projective principal congruence subgroup. In part (2), where every prime dividing N is inert in K, the additional projective level elements survive the global-sign condition of Proposition 3.8, and the full automorphism group is the strictly larger group $\mathrm{Bi}_K(2N)$. Thus, part (2) shows that $\mathrm{Bi}_K(2N)$, rather than $P\Gamma_K(2N)$, is the natural ambient congruence subgroup in the general higher-level theory.

The first part of the paper identifies the automorphism group arithmetically. The second part explains how this arithmetic group is realized on the surface itself. After projectivizing the positive cone, automorphisms of a $K3$ surface act by isometries of hyperbolic space, and these isometries are elliptic, parabolic, or loxodromic [13, Exercise 4.7-18]. In the present paper, the parabolic elements and involutions play central roles. Proposition 3.25 shows that the parabolic elements are precisely those preserving the genus-one fibrations, and that their unipotent actions are given by the corresponding Mordell–Weil translation groups. The distinguished involution is the deck involution of the double-cover model. It is also tied to the genus-one fibrations; see Lemma 4.10, Subsection 3.3, and Subsection 5, especially Lemmas 5.5–5.7.

At level 2, the surfaces $X_{K,2}$ arise from Kummer surfaces. Let $E_K \cong \mathbb{C}/\mathcal{O}_K$ be the elliptic curve with complex multiplication by $\mathcal{O}_K$, and set $A_K := E_K \times E_K$. The Kummer surface $\mathrm{Km}(A_K)$ contains a natural non-exceptional lattice isomorphic to $S_K(2)$. Deforming the Kummer surface while preserving this lattice produces the $S_K(2)$-polarized surface $X_{K,2}$. This construction is useful because it also produces explicit double-cover models and genus-one fibrations.

Theorem 4 (Proposition 4.6 and Corollary 4.14)**.** *Let $K = \mathbb{Q}(\sqrt{-d})$, and let $E_K \cong \mathbb{C}/\mathcal{O}_K$ be the elliptic curve with complex multiplication by $\mathcal{O}_K$. Set $A_K := E_K \times E_K$. Then the following hold.*

(1) *Choose a double-cover model $E_K = \{s^2 = B(x,y)\} \subset \mathbb{P}(1,1,2)$, where $B(x,y)$ is a binary quartic form. Then $X_{K,2}$ can be realized as a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form*
$$X_{K,2} = \left\{S^2 = B(x,y)B(z,w) + H_{4,4}(x,y;z,w)\right\} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1,$$
where $H_{4,4}$ is a bihomogeneous form of bidegree $(4,4)$ satisfying the corresponding Noether–Lefschetz condition. There is a covering involution $\iota\colon S \longmapsto -S$.

(2) *For $K = \mathbb{Q}(\sqrt{-2})$ and $K = \mathbb{Q}(\sqrt{-7})$, the group $\mathrm{Aut}(X_{K,2})$ is generated by Mordell–Weil translation groups associated with genus-one fibrations, together with the covering involution $\iota\colon S \longmapsto -S$.*

This involution is particularly significant. By Lemma 4.9, although there are infinitely many involutions, the complete-intersection models constructed below exhibit finite collections of geometrically natural deck involutions that generate the full automorphism group in the Gaussian and Eisenstein cases.

Theorem 5 (cf. Theorem 5.4 and 5.14)**.** *Let $K = \mathbb{Q}(i)$ or $K = \mathbb{Q}(\sqrt{-3})$. Then $X_{K,2}$ admits a complete-intersection model in a product of projective spaces, and $\mathrm{Aut}(X_{K,2})$ is generated by deck involutions.*

In the Gaussian case, these deck involutions act on hyperbolic three-space as edge half-turns in the regular ideal octahedral tiling. In the Eisenstein case, they act as edge half-turns in the regular ideal tetrahedral tiling (see Proposition 5.6). Hence, in these exceptional unit cases, the Bianchi group is visible not only through the lattice $S_K(2)$, but also through projective symmetries of the corresponding $K3$ surface.

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2. Preliminaries

Throughout this article, we work over $\mathbb{C}$. All $K3$ surfaces are assumed to be projective. The $K3$ lattice is defined by $\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Since $H^1(X, \mathcal{O}_X) = 0$, we identify $\mathrm{Pic}(X)$ with the Néron–Severi lattice $\mathrm{NS}(X)$ via the first Chern class. We shall use the following theorem of Morrison, which guarantees the existence of complex projective $K3$ surfaces with prescribed low-rank Picard lattice.

Theorem 2.1 ([11, cf. Corollary 2.9])**.** *Let L be an even lattice of signature $(1, \rho - 1)$ with $\rho \leq 10$. Then there exists a projective $K3$ surface X such that $\mathrm{Pic}(X) \cong L$.*

We recall the basic language of discriminant forms and primitive embeddings.

Definition 2.2. Let L be an integral, nondegenerate, even lattice. Its *dual lattice* is
$$L^\vee := \{x \in L \otimes \mathbb{Q} \mid (x,y) \in \mathbb{Z} \text{ for all } y \in L\}.$$
Because L is integral and even, the *discriminant group* $A_L := L^\vee/L$ carries a natural finite quadratic form:
$$q_L : A_L \longrightarrow \mathbb{Q}/2\mathbb{Z}, \qquad q_L(x + L) = (x,x) \pmod{2\mathbb{Z}}$$

Every isometry $g \in O(L)$ canonically induces an isometry $\bar{g} \in O(A_L, q_L)$.

Proposition 2.3 ([12, Proposition 1.6.1]). *Let Λ be an even unimodular lattice. If $S \subset \Lambda$ is a nondegenerate primitive sublattice and $T = S^\perp$ is its orthogonal complement, then Λ induces a canonical anti-isometry*

$$(A_S, q_S) \cong (A_T, -q_T)$$

between their discriminant forms.

Let S be a Picard lattice and let $T := S^\perp_{\Lambda_{K3}}$ be its orthogonal complement. The corresponding period domain is:

$$\Omega_S := \{[\omega] \in \mathbb{P}(T \otimes \mathbb{C}) \mid (\omega, \omega) = 0, \quad (\omega, \overline{\omega}) > 0\}$$

The next lemma says that, for a very general period in Ω_S, the transcendental Hodge structure has the smallest possible group of Hodge isometries.

Lemma 2.4. *Let S be a primitive hyperbolic sublattice of Λ_{K3}, and put $T := S^\perp_{\Lambda_{K3}}$. For a very general period point $[\omega] \in \Omega_S$, the group of Hodge isometries of the Hodge structure T_ω is $O_{\mathrm{Hdg}}(T_\omega) = \{\pm \mathrm{id}\}$.*

Proof. For $g \in O(T)$, define $\mathrm{Fix}(g) := \{[\omega] \in \Omega_S \mid g(\mathbb{C}\omega) = \mathbb{C}\omega\}$. This is a closed analytic subset of Ω_S, since the condition $g(\mathbb{C}\omega) = \mathbb{C}\omega$ is equivalent to $g\omega \wedge \omega = 0$ in $\bigwedge^2 T_\mathbb{C}$.

We claim that if $\mathrm{Fix}(g) = \Omega_S$, then $g = \pm\mathrm{id}$. Indeed, in that case g fixes every period line in Ω_S. Hence the induced projective transformation of $\mathbb{P}(T_\mathbb{C})$ is the identity on a nonempty open subset of the period quadric. Since this quadric spans $\mathbb{P}(T_\mathbb{C})$, the transformation is the identity on $\mathbb{P}(T_\mathbb{C})$. Therefore g acts as a scalar on $T_\mathbb{C}$. Since g is an isometry of the lattice T, this scalar must be ± 1. Thus $g = \pm\mathrm{id}$.

Consequently, for every $g \in O(T) \setminus \{\pm\mathrm{id}\}$, the locus $\mathrm{Fix}(g)$ is a proper closed analytic subset of Ω_S. Since $O(T)$ is countable, the union $\bigcup_{g \in O(T) \setminus \{\pm \mathrm{id}\}} \mathrm{Fix}(g)$ is a countable union of proper closed analytic subsets. A very general period point $[\omega] \in \Omega_S$ avoids this union.

Now let $h \in O_{\mathrm{Hdg}}(T_\omega)$. Since h is a Hodge isometry, it preserves the line $H^{2,0}(T_\omega) = \mathbb{C}\omega$. Hence $[\omega] \in \mathrm{Fix}(h)$. By the choice of $[\omega]$, this implies $h = \pm\mathrm{id}$. Conversely, both id and $-$id are Hodge isometries. Therefore $O_{\mathrm{Hdg}}(T_\omega) = \{\pm\mathrm{id}\}$. □

We use the following version of the Torelli theorem to compute the automorphism group of the $K3$ surfaces.

Theorem 2.5. [6, Chapter 15 Corollary 2.3] *Let X be a projective K3 surface and put $S = \mathrm{NS}(X)$ and $T = S^\perp \subset H^2(X, \mathbb{Z})$. Let $\Delta(S) = \{\delta \in S \mid \delta^2 = -2\}$, and let $W(S) \subset O^+(S)$ be the Weyl group generated by the reflections for $\Delta(S)$.*

Let $\Gamma_X \subset O^+(S)$ be the subgroup of isometries which are compatible with the Hodge structure on T, namely

$$\Gamma_X = \left\{ g \in O^+(S) \;\middle|\; \begin{array}{c} \text{there exists } h \in O_{\mathrm{Hdg}}(T) \text{ such that } g \oplus h \\ \text{extends to an isometry of } H^2(X, \mathbb{Z}) \end{array} \right\}.$$

Then $W(S) \triangleleft \Gamma_X$, and there is a natural exact sequence

$$1 \longrightarrow \ker\bigl(\mathrm{Aut}(X) \to O(S)\bigr) \longrightarrow \mathrm{Aut}(X) \longrightarrow \Gamma_X / W(S) \longrightarrow 1.$$

In particular, if the restriction homomorphism $\mathrm{Aut}(X) \longrightarrow O(S)$ is injective, then $\mathrm{Aut}(X) \cong \Gamma_X / W(S)$.

We use the following injectivity criterion for the automorphism groups.

Lemma 2.6. *Let X be a very general S-polarized projective $K3$ surface, and assume that the discriminant group A_S is not 2-elementary. Then the kernel of the natural representation $\mathrm{Aut}(X) \longrightarrow O(S)$ is trivial.*

Proof. Let $f \in \mathrm{Aut}(X)$ be in the kernel, so $f^*|_S = \mathrm{id}_S$. Then f^* restricts to a Hodge isometry of the transcendental lattice $T_X = S^\perp$. Since X is very general, Lemma 2.4 gives $f^*|_{T_X} = \pm \mathrm{id}_{T_X}$.

If $f^*|_{T_X} = +\mathrm{id}_{T_X}$, f^* is the identity on the full lattice $H^2(X, \mathbb{Z})$. By the Global Torelli theorem, $f = \mathrm{id}_X$, giving the trivial kernel $\{1\}$.

If $f^*|_{T_X} = -\mathrm{id}_{T_X}$, the split isometry $\mathrm{id}_S \perp (-\mathrm{id}_{T_X})$ extends to the unimodular overlattice $H^2(X, \mathbb{Z})$ if and only if the actions on the discriminant groups match. By Proposition 2.3, this requires $\mathrm{id}_{A_S} = -\mathrm{id}_{A_S}$, which means $2x = 0$ for all $x \in A_S$ i.e., A_S is 2-elementary. $\square$

3. Bianchi groups and automorphism groups

3.1. **Bianchi Groups.** Throughout this article, let $d > 0$ be a squarefree integer, and set $K = \mathbb{Q}(\sqrt{-d})$. Let $\mathcal{O}_K$ be the ring of integers of K, and let $-D$ be the fundamental discriminant of K. We consider the $\mathbb{Z}$-module

$$S_K := \mathrm{Herm}_2(\mathcal{O}_K) = \left\{ A = \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} : a, c \in \mathbb{Z},\ b \in \mathcal{O}_K \right\},$$

equipped with the quadratic form $q_d(A) := 2\det(A)$.

This module S_K can be naturally identified with a rank-four lattice equipped with an intersection form. To maintain notation independent of the choice of d, we will write $S_K \cong U \perp M_K$, where U is the standard hyperbolic plane and M_K is a rank-2 negative-definite lattice defined according to the fundamental discriminant:

(1) If $-D \equiv 0 \pmod 4$, then $M_K := \langle -2 \rangle \perp \langle -\frac{D}{2} \rangle$.

(2) If $-D \equiv 1 \pmod 4$, then $M_K := \Lambda_D$, where Λ_D is the lattice with Gram matrix $\begin{pmatrix} -2 & -1 \\ -1 & -\frac{1+D}{2} \end{pmatrix}$.

Definition 3.1. The *Bianchi group* Bi_K is defined as $\mathrm{PGL}_2(\mathcal{O}_K) = \mathrm{GL}_2(\mathcal{O}_K)/\mathcal{O}_K^\times$.

The Bianchi group acts on $S_K = \mathrm{Herm}_2(\mathcal{O}_K)$ by

$$[M] \cdot H := MHM^*, \qquad M \in \mathrm{GL}_2(\mathcal{O}_K), \quad H \in \mathrm{Herm}_2(\mathcal{O}_K),$$

where $M^* := {}^t\overline{M}$. This action is well-defined on $\mathrm{PGL}_2(\mathcal{O}_K)$, since multiplying M by a unit $\lambda \in \mathcal{O}_K^\times$ multiplies MHM^* by $\lambda\overline{\lambda} = 1$. Moreover, $\det(MHM^*) = \det(M)\overline{\det(M)}\det(H) = \det(H)$, because $\det(M) \in \mathcal{O}_K^\times$. Hence this gives a homomorphism

$$\mathrm{Bi}_K \longrightarrow O(S_K).$$

We choose the positive cone component C_K^+ to be the component containing the positive definite Hermitian matrices. Since positive definiteness is preserved by $H \longmapsto MHM^*$, the image of Bi_K lies in $O^+(S_K)$.

Remark 3.2. In the literature (e.g., Vinberg [16]), the *Bianchi group* Bi_K is often defined by the semidirect product of $\mathrm{PGL}_2(\mathcal{O}_K)$ and the non-trivial Galois automorphism σ of $K/\mathbb{Q}$. Geometrically, σ acts as an anti-holomorphic isometry. Lemma 3.10 shows that this Galois coset does not contribute automorphisms of the very general $K3$ surfaces considered here.

Vinberg [16] proved that $O^+(S_K) = \widehat{\mathrm{Bi}}_K \rtimes \langle \sigma \rangle$, where $\widehat{\mathrm{Bi}}_K$ denotes the orientation-preserving extended Bianchi group and σ is induced by complex conjugation. $\widehat{\mathrm{Bi}}_K$ acts by orientation-preserving isometries on $\mathbb{H}^3$ through the normalized action

$$A \longmapsto \frac{1}{|\det M|} MAM^*.$$

The normalized action realizes $\widehat{\mathrm{Bi}}_K$ as a subgroup of $O^+(S_K)$. $\widehat{\mathrm{Bi}}_K$ also contains projective classes of matrices $M \in \mathrm{GL}_2(K)$ satisfying $M\mathcal{O}_K^2 = \mathfrak{a}\,\mathcal{O}_K^2$ for a fractional ideal $\mathfrak{a}$ whose ideal class has order dividing two. By [2, Section 4], there is an exact sequence:

$$1 \longrightarrow \mathrm{Bi}_K \longrightarrow \widehat{\mathrm{Bi}}_K \longrightarrow \mathrm{Cl}(\mathcal{O}_K)[2] \longrightarrow 1.$$

Here $\mathrm{Cl}(\mathcal{O}_K)[2]$ denotes the 2-torsion subgroup of the ideal class group of $\mathcal{O}_K$. Consequently, using the order estimate of the 2-torsion group (for example see [3, Theorem 6.1]), we have the following statement.

Proposition 3.3. *Let $K = \mathbb{Q}(\sqrt{-d})$ with d square-free. Then Bi_K is a normal subgroup of $\widehat{\mathrm{Bi}}_K$, and their index is given by $[\widehat{\mathrm{Bi}}_K : \mathrm{Bi}_K] = 2^{\omega(-D)-1}$ where $\omega(-D)$ is the number of distinct primes dividing the discriminant $-D$.*

Example 3.4. Let $K = \mathbb{Q}(\sqrt{-p})$, $p \equiv 1 \pmod 4$, and write $\tau = \sqrt{-p}$. Then $\mathcal{O}_K = \mathbb{Z}[\tau]$, $\mathrm{disc}(K) = -4p$. By the preceding proposition, we have $[\widehat{\mathrm{Bi}}_K : \mathrm{Bi}_K] = 2$. The nontrivial class is represented by the ramified prime ideal $\mathfrak{p}_2 = (2, 1+\tau)$, $\mathfrak{p}_2^2 = (2)$. Set

$$W_2 = \begin{pmatrix} 1+\tau & 2 \\ 2s & 1-\tau \end{pmatrix}, \qquad s = \frac{p-1}{4}.$$

Then $\det(W_2) = 2$ and a direct ideal calculation gives $W_2\mathcal{O}_K^2 = \mathfrak{p}_2\mathcal{O}_K^2$. Since $\mathfrak{p}_2$ is nonprincipal, no scalar multiple of W_2 belongs to $\mathrm{GL}_2(\mathcal{O}_K)$. Hence

$$[W_2] \in \widehat{\mathrm{Bi}}_K \setminus \mathrm{Bi}_K,$$

and $[W_2]$ represents the unique nontrivial coset of $\widehat{\mathrm{Bi}}_K / \mathrm{Bi}_K$. Nevertheless, the normalized action

$$A \longmapsto \frac{1}{2} W_2 A W_2^*$$

preserves the integral Hermitian lattice S_K. Indeed, the entries of W_2 lie in $\mathfrak{p}_2$, and therefore

$$W_2 A W_2^* \in \mathfrak{p}_2\overline{\mathfrak{p}}_2 \,\mathrm{Herm}_2(\mathcal{O}_K) = 2\,\mathrm{Herm}_2(\mathcal{O}_K).$$

Thus the factor $1/2$ restores integrality, and the resulting transformation preserves the determinant quadratic form.

We compute the discriminant group $A_{S_K(2N)}$.

Lemma 3.5. *Let K be an imaginary quadratic field of discriminant $-D$, $D > 0$. For an integer $N \geq 1$, consider the scaled lattice $S_K(2N)$. Let*

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P(z) = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}.$$

On $S_K(2N)$, the elements e, f span a scaled copy of U, and the classes

$$\frac{e}{2N}, \qquad \frac{f}{2N}$$

generate the $U(2N)^\vee/U(2N)$-part of the discriminant group. The remaining generators are as follows:

(1) *If $D \equiv 0 \pmod 4$, let $\tau = \frac{\sqrt{-D}}{2}$. The off-diagonal block is generated by $P(1)$ and $P(\tau)$, yielding the corresponding dual generators:*

$$\frac{P(1)}{4N}, \qquad \frac{P(\tau)}{DN}.$$

(2) *If* $D \equiv 3 \pmod 4$, *let* $\tau = \frac{1+\sqrt{-D}}{2}$. *In the basis* $P(1), P(\tau)$, *the corresponding dual generators are:*

$$\frac{(D+1)P(1) - 2P(\tau)}{4DN}, \qquad \frac{-2P(1) + 4P(\tau)}{4DN}.$$

Proof. The bilinear form on $S_K(2N)$ is $(X,Y)_{2N} = 2N(\det(X+Y) - \det(X) - \det(Y))$. Hence $(e,e)_{2N} = (f,f)_{2N} = 0$ and $(e,f)_{2N} = 2N$. Therefore e, f span a scaled hyperbolic plane, and its dual is directly generated by $e/(2N)$ and $f/(2N)$.

For the off-diagonal matrices, $(P(z), P(w))_{2N} = -2N \operatorname{Tr}_{K/\mathbb{Q}}(z\bar{w})$.

If $D \equiv 0 \pmod 4$, with $\tau = \sqrt{-D}/2$, the Gram matrix of $P(1), P(\tau)$ scaled by N is

$$\begin{pmatrix} -4N & 0 \\ 0 & -DN \end{pmatrix}.$$

Inverting this matrix immediately yields the dual generators $P(1)/(4N)$ and $P(\tau)/(DN)$.

If $D \equiv 3 \pmod 4$, with $\tau = (1+\sqrt{-D})/2$, the Gram matrix of $P(1), P(\tau)$ scaled by N is

$$\Lambda_{D,N} = N \begin{pmatrix} -4 & -2 \\ -2 & -(D+1) \end{pmatrix} \qquad \Lambda_{D,N}^{-1} = \frac{1}{4DN} \begin{pmatrix} -(D+1) & 2 \\ 2 & -4 \end{pmatrix}.$$

Thus the dual basis is generated, up to signs, by

$$\frac{(D+1)P(1) - 2P(\tau)}{4DN}, \qquad \frac{-2P(1) + 4P(\tau)}{4DN}.$$

This proves the claim. $\square$

3.2. **Automorphism groups of $K3$ surfaces.** For an integer $m \geq 1$, we define the congruence subgroups of level m as follows:

Definition 3.6. Let $m \geq 1$ be an integer. The *congruence subgroup* of level m is defined as

$$\mathrm{Bi}_K(m) := \ker(\mathrm{PGL}_2(\mathcal{O}_K) \longrightarrow \mathrm{PGL}_2(\mathcal{O}_K/m\mathcal{O}_K)).$$

We define the *principal congruence subgroup* of level m as

$$P\Gamma_K(m) := \left\{ [M] \in \mathrm{PGL}_2(\mathcal{O}_K) \;\middle|\; \begin{array}{c} \text{some representative } M \in \mathrm{GL}_2(\mathcal{O}_K) \\ \text{satisfies } M \equiv I_2 \pmod{m\mathcal{O}_K} \end{array} \right\}.$$

We first compare the two congruence subgroups introduced above.

Proposition 3.7. *Let* K *be an imaginary quadratic field. The index* $[\mathrm{Bi}_K(2) : P\Gamma_K(2)]$ *and the structure of* $\mathrm{Bi}_K(2)$ *depend on the factorization of* 2 *in* $\mathcal{O}_K$ *as follows:*

(1) *If* 2 *is unramified, then* $\mathrm{Bi}_K(2) = P\Gamma_K(2)$.
(2) *If* 2 *ramifies, then* $[\mathrm{Bi}_K(2) : P\Gamma_K(2)] \leq 2$.

Proof. Let $R_K = \mathcal{O}_K/2\mathcal{O}_K$. For any $[M] \in \mathrm{Bi}_K(2)$, its reduction modulo 2 is a scalar matrix $\overline{M} = aI_2$ for some $a \in R_K^\times$. Scaling the representative M by a global unit $u \in \mathcal{O}_K^\times$ multiplies a by $\overline{u}$, yielding a well-defined, injective homomorphism

$$\mathrm{Bi}_K(2)/P\Gamma_K(2) \hookrightarrow R_K^\times/\overline{\mathcal{O}_K^\times}.$$

Suppose first that 2 is unramified. Then either 2 splits or 2 is inert. If 2 splits, $R_K \cong \mathbb{F}_2 \times \mathbb{F}_2$, so $R_K^\times = \{1\}$. Thus $a = 1$, and $[M] \in P\Gamma_K(2)$. If 2 is inert, $R_K \cong \mathbb{F}_4$, and $a^2 = \overline{\det(M)}$. If $K \neq \mathbb{Q}(\sqrt{-3})$, then $\mathcal{O}_K^\times = \{\pm 1\}$, so $\overline{\det(M)} = 1$. Since $\mathbb{F}_4^\times$ has order 3, the equality $a^2 = 1$ implies $a = 1$. If $K = \mathbb{Q}(\sqrt{-3})$, the reduction homomorphism $\mathcal{O}_K^\times \to \mathbb{F}_4^\times$ is surjective. We may therefore choose $u \in \mathcal{O}_K^\times$ such that $\overline{u} = a^{-1}$.

Then $uM \equiv I_2 \pmod{2\mathcal{O}_K}$, meaning uM represents the same element in $\mathrm{PGL}_2(\mathcal{O}_K)$. Thus $[M] \in P\Gamma_K(2)$. This establishes $\mathrm{Bi}_K(2) = P\Gamma_K(2)$ in the unramified case.

Finally, suppose that 2 ramifies. In this case, the local ring structure gives $R_K \cong \mathbb{F}_2[\epsilon]/(\epsilon^2)$, so $|R_K^\times| = 2$. The injective homomorphism therefore bounds the index to $[\mathrm{Bi}_K(2) : P\Gamma_K(2)] \leq 2$. □

We now turn to $K3$ surfaces. Throughout this subsection, N is an odd positive integer. Let $X_{K,2N}$ be a very general K3 surface with $\mathrm{NS}(X_{K,2N}) \cong S_K(2N)$.

The guiding principle is that a Bianchi isometry is induced by an automorphism of $X_{K,2N}$ only when its action on the discriminant group matches the action of a Hodge isometry on the transcendental lattice. Since $X_{K,2N}$ is very general, the only Hodge isometries of the transcendental lattice are $\pm\,\mathrm{id}$. Thus, the $K3$ condition is a *global sign condition* on the discriminant group.

Proposition 3.8. *Let N be an odd positive integer, and let X be a very general $K3$ surface with $S := \mathrm{NS}(X) \simeq S_K(2N)$. An isometry $g \in O(S)$ is induced by an automorphism of X if and only if g preserves the ample cone and its induced action on the discriminant group A_S is multiplication by a single sign ± 1.*

Proof. If $g = f^*|_S$ for $f \in \mathrm{Aut}(X)$, then g trivially preserves the ample cone. Since X is very general, $f^*|_T = \pm\,\mathrm{id}_T$ by Lemma 2.4. Compatibility via the canonical anti-isometry $A_S \to A_T$ of discriminant forms forces g to act on A_S by the same sign ± 1.

Conversely, if g preserves the ample cone and acts on A_S by a sign $\sigma \in \{\pm 1\}$, setting $h := \sigma\,\mathrm{id}_T \in O(T)$ ensures compatibility on the discriminant groups. Thus, $g \oplus h$ extends to a Hodge isometry of $H^2(X, \mathbb{Z})$. Since g preserves the ample cone, the global Torelli theorem implies this isometry is induced by an automorphism of X. □

Corollary 3.9. *Let X be as above. An element $g \in \mathrm{Bi}_K$ is induced by an automorphism of X if and only if $g \in \mathrm{Bi}_K(2N)$ and the induced action of g on A_S is multiplication by a single sign ± 1.*

Proof. Let $g \in \mathrm{Bi}_K$ be represented by a matrix $M \in \mathrm{GL}_2(\mathcal{O}_K)$. The Bianchi action on S_K is given by

$$H \longmapsto MHM^*.$$

If H is positive definite Hermitian, MHM^* is also positive definite Hermitian. Hence, g preserves the connected component C_S^+ of the positive cone containing I_2. Since $S_K(2N)$ has no (-2)-classes, C_S^+ is exactly the ample cone, meaning g always preserves it. By Proposition 3.8, g is induced by an automorphism of X if and only if its action on A_S is a sign ± 1.

It remains to show that this sign action forces $g \in \mathrm{Bi}_K(2N)$ (the converse is trivial). Inside A_S we have the level subgroup $S_K/(2N)S_K \hookrightarrow A_S$ given by $x \mapsto \frac{x}{2N}$. Under the Hermitian model, this subgroup is naturally identified with $\mathrm{Herm}_2(R_{2N})$, where $R_{2N} := \mathcal{O}_K/2N\mathcal{O}_K$.

Let $\overline{M}$ denote the reduction of M modulo $2N$. Since g acts on this visible level subgroup by some sign $\epsilon \in \{\pm 1\}$, we have $\overline{M}H\overline{M}^* = \epsilon H$ for every $H \in \mathrm{Herm}_2(R_{2N})$.

Applying this identity to the standard basis elements of $\mathrm{Herm}_2(R_{2N})$:

- For $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we get $aa^* = \epsilon$ (so $a \in R_{2N}^\times$) and $ca^* = 0$ (so $c = 0$).
- For $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we get $b = 0$ and $dd^* = \epsilon$.
- For $P(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we get $ad^* = \epsilon$.

Together with $aa^* = \epsilon$, the last condition forces $d = a$. Hence $\overline{M} = aI_2$. The reduction of g modulo $2N$ is therefore projectively trivial, concluding that $g \in \mathrm{Bi}_K(2N)$. □

Now using the expression of Lemma 3.5, we will justify why no Galois-conjugation coset element is induced by an automorphism.

Lemma 3.10. *Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, and let $N \geq 1$ be odd. Let $X = X_{K,2N}$ be a very general $S_K(2N)$-polarized K3 surface. Let σ be the nontrivial Galois automorphism of $K/\mathbb{Q}$, acting on $S_K = \mathrm{Herm}_2(\mathcal{O}_K)$ by $\sigma(P(z)) = P(\overline{z})$. Then no element of the coset $\mathrm{Bi}_K\,\sigma$ is induced by an automorphism of X.*

Proof. Suppose, for a contradiction, that an element of the form $M\sigma$, with $M \in \mathrm{GL}_2(\mathcal{O}_K)$, is induced by an automorphism of X. Since X is very general, the only Hodge isometries of the transcendental lattice are $\pm\,\mathrm{id}$. Hence the action of $M\sigma$ on the discriminant group $A_{S_K(2N)}$ must be multiplication by a single sign $\varepsilon \in \{\pm 1\}$.

In particular, on the visible level subgroup $S_K/(2N)S_K \subset A_{S_K(2N)}$, the induced action must also be multiplication by ε. Thus, for every $H \in \mathrm{Herm}_2(\mathcal{O}_K/2N\mathcal{O}_K)$, we have $M\sigma(H)M^* \equiv \varepsilon H \pmod{2N\mathcal{O}_K}$.

Applying this congruence to e, f, and $P(1)$, which are fixed by σ, gives the same calculation as in Proposition 3.8: there exists a unit $u \in (\mathcal{O}_K/2N\mathcal{O}_K)^\times$ such that $M \equiv uI_2 \pmod{2N\mathcal{O}_K}$ and $u\overline{u} \equiv \varepsilon \pmod{2N}$.

Now choose an integral basis $1, \tau$ of $\mathcal{O}_K$. We distinguish the two cases.

First suppose that the fundamental discriminant is odd, so that $\tau = \frac{1+\sqrt{-D}}{2}$ and $\overline{\tau} = 1 - \tau$. Applying the above congruence to $P(\tau)$, we obtain $M\sigma(P(\tau))M^* \equiv u\overline{u}P(\overline{\tau}) \equiv \varepsilon P(1-\tau) \pmod{2N}$. But the global sign condition requires this to be congruent to $\varepsilon P(\tau)$ modulo $2N$. Hence $P(1-2\tau) \equiv 0 \pmod{2N}$. This is impossible, because $1 - 2\tau \notin 2N\mathcal{O}_K$.

Next suppose that the fundamental discriminant is even. Write $\tau = \frac{\sqrt{-D}}{2}$ and $\overline{\tau} = -\tau$. If $N > 1$, the same argument gives $P(-\tau) \equiv P(\tau) \pmod{2N}$, or equivalently $2\tau \in 2N\mathcal{O}_K$, which is impossible.

It remains to treat the case $N = 1$. In this case, the level subgroup alone does not give a contradiction, so we use the two-primary discriminant classes. The class $P(1)/4 \in A_{S_K(2)}$ gives $MP(1)M^* \equiv \varepsilon P(1) \pmod{4S_K}$. Writing $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and using $M \equiv uI_2 \pmod{2\mathcal{O}_K}$, we get $a\overline{d} + b\overline{c} \equiv \varepsilon \pmod{4\mathcal{O}_K}$. Since $b, c \in 2\mathcal{O}_K$, this implies $a\overline{d} \equiv \varepsilon \pmod{4\mathcal{O}_K}$.

On the other hand, the class $P(\tau)/D \in A_{S_K(2)}$ gives $M\sigma(P(\tau))M^* \equiv \varepsilon P(\tau) \pmod{DS_K}$. Since $\sigma(P(\tau)) = P(-\tau)$, the off-diagonal entry of $M\sigma(P(\tau))M^*$ is congruent to $-\tau a\overline{d} \equiv -\varepsilon\tau \pmod{4\tau\mathcal{O}_K}$. The global sign condition requires this to be congruent to $\varepsilon\tau$ modulo $D\mathcal{O}_K$. Hence $2\tau \in D\mathcal{O}_K + 4\tau\mathcal{O}_K$. Since $D = -4\tau^2$, we have $D\mathcal{O}_K \subset 4\tau\mathcal{O}_K$. Therefore $2\tau \in 4\tau\mathcal{O}_K$, which forces $1 \in 2\mathcal{O}_K$, a contradiction.

Thus no element of $\mathrm{Bi}_K\,\sigma$ can be induced by an automorphism of X. ☐

We now compute the necessary local characters. The discriminant group $A_{S(2N)} = \frac{1}{2N}S^\vee/S$ sits in an exact sequence:

$$0 \longrightarrow S/(2N)S \longrightarrow A_{S(2N)} \longrightarrow A_S \longrightarrow 0.$$

The global sign condition thus has two sources: the level part $S/(2N)S$, and the original discriminant group A_S. First consider the primes dividing N. For an integer $m \geq 1$, put $\mathcal{N}_m := \{u\overline{u} \mid u \in R_m^\times\} \subset (\mathbb{Z}/m\mathbb{Z})^\times$.

Lemma 3.11. *Let $g \in \mathrm{Bi}_K(2N)$, and let $p^e \| N$ meaning that p^e exactly divides N. Choose a representative $M \in \mathrm{GL}_2(\mathcal{O}_K)$. Then there exists $u_{p^e} \in (\mathcal{O}_K/p^e\mathcal{O}_K)^\times$ such that $M \equiv u_{p^e}I_2 \pmod{p^e\mathcal{O}_K}$. The element $\alpha_{p^e}(g) := u_{p^e}\overline{u}_{p^e} \in \mathcal{N}_{p^e}$ is independent of the representative M. Moreover, on $S/p^eS \cong \mathrm{Herm}_2(\mathcal{O}_K/p^e\mathcal{O}_K)$, g acts by scalar multiplication with $\alpha_{p^e}(g)$.*

Proof. Since $g \in \mathrm{Bi}_K(2N)$, its reduction modulo $p^e\mathcal{O}_K$ is projectively scalar, so $M \equiv u_{p^e}I_2 \pmod{p^e\mathcal{O}_K}$. For any $H \in \mathrm{Herm}_2(\mathcal{O}_K/p^e\mathcal{O}_K)$, $MHM^* \equiv (u_{p^e}I_2)H(\overline{u}_{p^e}I_2) = u_{p^e}\overline{u}_{p^e}H \pmod{p^e}$. Replacing M by λM for $\lambda \in \mathcal{O}_K^\times$ scales u_{p^e} by λ. Since $\lambda\overline{\lambda} = 1$, the norm $u_{p^e}\overline{u}_{p^e}$ is well-defined. ☐

Next, consider A_S. Its odd primary part only appears at odd primes ramified in K, where the action is controlled by the determinant.

Lemma 3.12. *Let $\ell \neq 2$ be a prime dividing D, and write $\ell\mathcal{O}_K = \mathfrak{p}^2$. Let $\delta = \sqrt{-D}$. Then the ℓ-primary part $A_{S_K,\ell}$ is generated by $u_\ell := \frac{1}{\ell}P(\delta) \in S_K^\vee/S_K$. For $g = [M] \in \mathrm{Bi}_K$, with $M \in \mathrm{GL}_2(\mathcal{O}_K)$, the induced action of g on $A_{S_K,\ell}$ is multiplication by $\varepsilon_\ell(g) := \det(M) \bmod \mathfrak{p} \in (\mathcal{O}_K/\mathfrak{p})^\times \cong \mathbb{F}_\ell^\times$. Moreover, this scalar is a sign: $\varepsilon_\ell(g) \in \{\pm 1\}$.*

Proof. Write $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. A direct calculation gives $MP(\delta)M^* = \mathrm{Tr}_{K/\mathbb{Q}}(a\delta\overline{b})e + \mathrm{Tr}_{K/\mathbb{Q}}(c\delta\overline{d})f + P(\delta(a\overline{d} - b\overline{c}))$. For every $x \in \mathcal{O}_K$, we have $\mathrm{Tr}_{K/\mathbb{Q}}(\delta x) = \delta(x - \overline{x}) \in D\mathbb{Z} \subset \ell\mathbb{Z}$. Hence the diagonal terms vanish modulo ℓS_K. Since ℓ is ramified, complex conjugation acts trivially on $\mathcal{O}_K/\mathfrak{p}$. Therefore $a\overline{d} - b\overline{c} \equiv ad - bc = \det(M) \pmod{\mathfrak{p}}$. Because $\delta \in \mathfrak{p}$, this implies $\delta\big(a\overline{d} - b\overline{c} - \det(M)\big) \in \mathfrak{p}^2 = \ell\mathcal{O}_K$. Thus $MP(\delta)M^* \equiv P(\delta\det(M)) \pmod{\ell S_K}$. Dividing by ℓ, we obtain $g(u_\ell) = \varepsilon_\ell(g)u_\ell$, where $\varepsilon_\ell(g) = \det(M) \bmod \mathfrak{p}$.

It remains to check that $\varepsilon_\ell(g)$ is well-defined and is a sign. If M is replaced by λM, with $\lambda \in \mathcal{O}_K^\times$, then $\det(\lambda M) = \lambda^2\det(M)$. For an odd ramified prime ℓ, the reduction of every global unit satisfies $\lambda^2 \equiv 1 \pmod{\mathfrak{p}}$. Indeed, if $\mathcal{O}_K^\times = \{\pm 1\}$, this is immediate. The only imaginary quadratic field with extra units and an odd ramified prime is $K = \mathbb{Q}(\sqrt{-3})$, where $\ell = 3$ and $(\mathcal{O}_K/\mathfrak{p})^\times \cong \mathbb{F}_3^\times = \{\pm 1\}$. Therefore the scalar $\varepsilon_\ell(g)$ is independent of the representative M. Finally, since $M \in \mathrm{GL}_2(\mathcal{O}_K)$, we have $\det(M) \in \mathcal{O}_K^\times$. By the preceding discussion, its reduction modulo $\mathfrak{p}$ is a sign. Hence $\varepsilon_\ell(g) \in \{\pm 1\}$. □

We now package these local scalars. Define

$$R_N := \left(\prod_{\ell \mid D, \ell \neq 2} \{\pm 1\}\right) \times \left(\prod_{p^e \| N} \mathcal{N}_{p^e}\right),$$

and let $\Psi_N \colon \mathrm{Bi}_K(2N) \longrightarrow R_N$ be given by $\Psi_N(g) := \big((\epsilon_\ell(g))_{\ell \mid D, \ell \neq 2}, (\alpha_{p^e}(g))_{p^e \| N}\big)$. The diagonal sign subgroup is $\Delta_N := \big\{\big((\sigma), (\sigma)\big) \mid \sigma \in \{\pm 1\}, \sigma \in \mathcal{N}_{p^e}\big\}$.

Definition 3.13. Let $\mathcal{C}_N \subset \mathrm{Bi}_K(2N)$ be the subgroup of elements g for which there exists a single sign $\sigma \in \{\pm 1\}$ such that:

(1) $\alpha_{p^e}(g) = \sigma$ for every $p^e \| N$;
(2) $\epsilon_\ell(g) = \sigma$ for every odd prime $\ell \mid D$;
(3) If $2 \mid D$, the induced action of g on $A_{S_K(2N),2}$ is multiplication by σ.

Theorem 3.14. *Let K be an imaginary quadratic field with fundamental discriminant $-D$. Let N be an odd positive integer, and assume that no prime divisor of N ramifies in K. Let $X = X_{K,2N}$ be very general with $\mathrm{NS}(X) \cong S_K(2N)$. Then $\mathrm{Bi}_K \cap \mathrm{Aut}(X) = \mathcal{C}_N$. In particular, if $2 \nmid D$, then $\mathrm{Bi}_K \cap \mathrm{Aut}(X) = \Psi_N^{-1}(\Delta_N)$.*

Proof. By Proposition 3.8, $g \in \mathrm{Bi}_K$ is induced by an automorphism of X if and only if $g \in \mathrm{Bi}_K(2N)$ and g acts on $A_{S_K(2N)}$ by a single sign $\sigma \in \{\pm 1\}$. For $p^e \parallel N$, p is unramified in K, so $S_K \otimes \mathbb{Z}_p$ is unimodular; g acts on $A_{S_K(2N),p}$ by $\alpha_{p^e}(g)$, forcing $\alpha_{p^e}(g) = \sigma$. For an odd prime $\ell \mid D$, $\ell \nmid N$, so $A_{S_K(2N),\ell} \cong A_{S_K,\ell}$, forcing $\epsilon_\ell(g) = \sigma$. If $2 \mid D$, the condition on $A_{S_K(2N),2}$ completes the definition of $\mathcal{C}_N$. If $2 \nmid D$, $S_K \otimes \mathbb{Z}_2$ is unimodular; since $g \in \mathrm{Bi}_K(2N)$, its 2-primary action is trivial, and $+1 \equiv -1 \pmod 2$ poses no obstruction. The condition then reduces exactly to $\Psi_N(g) \in \Delta_N$. □

Corollary 3.15. *Let K be an imaginary quadratic field with fundamental discriminant $-D$ such that $2 \nmid D$. Let N be an odd positive integer, and assume that no prime divisor of N ramifies in K. Let $X = X_{K,2N}$ be very general with $\mathrm{NS}(X) \cong S_K(2N)$. Then*

$$[\mathrm{Bi}_K(2N) : \mathrm{Bi}_K \cap \mathrm{Aut}(X)] = \frac{|\operatorname{im}\Psi_N|}{|\operatorname{im}\Psi_N \cap \Delta_N|}.$$

Moreover,

$$[\mathrm{Bi}_K(2N) : \mathrm{Bi}_K \cap \mathrm{Aut}(X)] \le 2^{\omega(N)},$$

where $\omega(N)$ denotes the number of distinct prime divisors of N.

Proof. By Theorem 3.14, $\mathrm{Bi}_K \cap \mathrm{Aut}(X) = \Psi_N^{-1}(\Delta_N)$. The First Isomorphism Theorem therefore gives

$$[\mathrm{Bi}_K(2N) : \mathrm{Bi}_K \cap \mathrm{Aut}(X)] = \frac{|\operatorname{im}\Psi_N|}{|\operatorname{im}\Psi_N \cap \Delta_N|}.$$

We now describe a subgroup containing $\operatorname{im}\Psi_N$. For $g = [M] \in \mathrm{Bi}_K(2N)$ and $p^e \| N$, choose $u_{p^e} \in (\mathcal{O}_K/p^e\mathcal{O}_K)^\times$ such that $M \equiv u_{p^e} I_2 \pmod{p^e\mathcal{O}_K}$. Taking determinants gives $u_{p^e}^2 \equiv \det(M) \pmod{p^e\mathcal{O}_K}$. Hence

$$\alpha_{p^e}(g)^2 = (u_{p^e}\overline{u}_{p^e})^2 \equiv \det(M)\overline{\det(M)} = 1 \pmod{p^e}.$$

Since p is odd, the only solutions of $x^2 = 1$ in $(\mathbb{Z}/p^e\mathbb{Z})^\times$ are ± 1. Therefore $\alpha_{p^e}(g) \in \{\pm 1\}$.

On the other hand, Lemma 3.12 gives $\epsilon_\ell(g) = \det(M) \pmod{\mathfrak{p}_\ell}$ for every odd prime $\ell \mid D$. If $K \ne \mathbb{Q}(\sqrt{-3})$, then $\mathcal{O}_K^\times = \{\pm 1\}$, so all the values $\epsilon_\ell(g)$ are equal to a single sign. If $K = \mathbb{Q}(\sqrt{-3})$, then $D = 3$, so there is only one odd ramified prime and the same conclusion is automatic.

It follows that $\operatorname{im}\Psi_N \subset H_N$, where

$$H_N := \left\{ \left((\eta)_{\substack{\ell \mid D \\ \ell \ne 2}}, (a_{p^e})_{p^e \| N} \right) \,\middle|\, \eta, a_{p^e} \in \{\pm 1\} \right\}.$$

The diagonal sign subgroup Δ_N is contained in H_N, and $[H_N : \Delta_N] = 2^{\omega(N)}$. Writing $I_N := \operatorname{im}\Psi_N$, we obtain

$$\frac{|I_N|}{|I_N \cap \Delta_N|} = [I_N\Delta_N : \Delta_N] \le [H_N : \Delta_N] = 2^{\omega(N)}.$$

This proves the assertion. □

Corollary 3.16. *Let $p \equiv 3 \pmod 4$ be a prime, and let $K = \mathbb{Q}(\sqrt{-p})$. Let $X_{K,2}$ be very general with $\mathrm{NS}(X_{K,2}) \cong S_K(2)$. Then $\mathrm{Aut}(X_{K,2}) \cong \mathrm{Bi}_K(2) \cong P\Gamma_K(2)$.*

Proof. Because $p \equiv 3 \pmod 4$, the fundamental discriminant of K is exactly $-p$, which is an odd prime (so $2 \nmid p$). Applying Corollary 3.15 with $N = 1$, we get $\mathrm{Bi}_K(2) = \mathrm{Bi}_K \cap \mathrm{Aut}(X)$.

Furthermore, because the fundamental discriminant $-p$ has only one prime divisor, genus theory implies that the 2-rank of the ideal class group $\mathrm{Cl}(K)$ is 0. Thus, $\mathrm{Cl}(K)$ has odd order, meaning the quotient $\mathrm{Cl}(K)/\mathrm{Cl}(K)^2$ is trivial. Consequently, the extended Bianchi group is simply $O^+(S_K) = \mathrm{Bi}_K \rtimes \langle\sigma\rangle$. By Lemma 3.10, the coset $\mathrm{Bi}_K\,\sigma$ is not geometrically induced, ensuring $\mathrm{Aut}(X_{K,2}) \subseteq \mathrm{Bi}_K$. Injectivity of the lattice representation gives $\mathrm{Aut}(X_{K,2}) = \mathrm{Bi}_K(2)$. Combine with Proposition 3.7, we also got $\mathrm{Aut}(X_{K,2}) = P\Gamma(2)$. □

We now specialize to very general $S_K(2)$-polarized K3 surfaces where the prime 2 ramifies in the imaginary quadratic field K. The odd level characters disappear, and by Proposition 3.8, we need only determine which isometries satisfy the global sign condition on the discriminant group $A_{S_K(2)}$. We examine three distinct settings where 2 ramifies: $K = \mathbb{Q}(i)$, $K = \mathbb{Q}(\sqrt{-2})$, and the infinite family $K = \mathbb{Q}(\sqrt{-p})$ for primes $p \equiv 1 \pmod 4$.

Theorem 3.17. *Let $K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-2})\}$, or let $K = \mathbb{Q}(\sqrt{-p})$ for an odd prime $p \equiv 1 \pmod 4$. Let $X_{K,2}$ be a very general $S_K(2)$-polarized K3 surface. The natural representation $\rho\colon \mathrm{Aut}(X_{K,2}) \to O(S_K(2))$ is injective, and its image is precisely the principal congruence subgroup, so that: $\mathrm{Aut}(X_{K,2}) \cong P\Gamma_K(2)$.*

We evaluate which isometries of the lattice $S_K(2)$ are induced by geometric automorphisms. The proof of the theorem relies on the following three lemmas.

Lemma 3.18. *Every element of $P\Gamma_K(2)$ acts on the discriminant group $A_{S_K(2)}$ by a global sign. Consequently, $P\Gamma_K(2) \subseteq \mathrm{Aut}(X_{K,2})$.*

Proof. Let $[M] \in P\Gamma_K(2)$, and choose a representative $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_K)$ such that $M \equiv I_2 \pmod{2\mathcal{O}_K}$. This congruence implies $\det(M) \equiv 1 \pmod{2\mathcal{O}_K}$. For $K \in \{\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-p})\}$, the global units are $\{\pm 1\}$, which both reduce to 1 modulo 2. For $K = \mathbb{Q}(i)$, the global units are $\{\pm 1, \pm i\}$; since $\pm i \not\equiv 1 \pmod{2\mathbb{Z}[i]}$, the determinant congruence forces $\det(M) \in \{\pm 1\}$. Thus, in all three cases, $\det(M) \in \{\pm 1\}$; we set $\delta := \det(M)$.

We verify the action $H \mapsto MHM^*$ on the generators of $A_{S_K(2)}$. Since $M \equiv I_2 \pmod{2\mathcal{O}_K}$, the action of M on $S_K/2S_K$ is trivial. Therefore the classes $e/2$ and $f/2$ are fixed in $A_{S_K(2)}$. Since they are 2-torsion, multiplication by $+1$ and by -1 agree on them. For the discriminant generators, we separate the cases:

- *Generator $P(1)$ (all fields):* The generator is $P(1)/4$, so we work modulo $4S_K$. The off-diagonal entry of $MP(1)M^*$ is $a\overline{d} + b\overline{c}$. We evaluate the difference from δ:
$$a\overline{d} + b\overline{c} - \delta = a\overline{d} + b\overline{c} - (ad - bc) = a(\overline{d} - d) + b(\overline{c} + c).$$
Because $a, d \equiv 1 \pmod{2\mathcal{O}_K}$ and $b, c \in 2\mathcal{O}_K$, both terms are divisible by 4. The diagonal entries $a\overline{b} + b\overline{a}$ and $c\overline{d} + d\overline{c}$ are multiples of 4 by the same parity argument. Thus, $MP(1)M^* \equiv \delta P(1) \pmod{4S_K}$.
- *Generator $P(i)$ ($K = \mathbb{Q}(i)$):* Working modulo $4S_K$, the off-diagonal entry of $MP(i)M^*$ is $i(a\overline{d} - b\overline{c})$. We check the difference:
$$i(a\overline{d} - b\overline{c}) - i\delta = i\left(a(\overline{d} - d) - b(\overline{c} - c)\right).$$
The term in parentheses is divisible by 4, and the diagonal entries are similarly divisible by 4. Thus, $MP(i)M^* \equiv \delta P(i) \pmod{4S_K}$.
- *Generator $P(\tau)$ ($K \in \{\mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-p})\}$):* The generator is $P(\tau)/D$, working modulo DS_K (where $D = 8$ or $D = 4p$). The off-diagonal entry of $MP(\tau)M^*$ is $\tau(a\overline{d} - b\overline{c})$. The difference is:
$$\tau(a\overline{d} - b\overline{c}) - \delta\tau = \tau\left(a(\overline{d} - d) - b(\overline{c} - c)\right).$$
The parenthetical term is divisible by 4τ, making the entire expression divisible by $4\tau^2 = -D$. The diagonal entries have the form $\tau(a\overline{b} - b\overline{a})$ and $\tau(c\overline{d} - d\overline{c})$. Since $b, c \in 2\mathcal{O}_K$ and $a, d \equiv 1 \pmod{2\mathcal{O}_K}$, the expressions $a\overline{b} - b\overline{a}$ and $c\overline{d} - d\overline{c}$ are divisible by 4τ. Multiplying by τ, the diagonal entries are divisible by $4\tau^2 = -D$. Hence the diagonal entries vanish modulo DS_K. Thus, $MP(\tau)M^* \equiv \delta P(\tau) \pmod{DS_K}$.

Consequently, every element of $P\Gamma_K(2)$ acts on $A_{S_K(2)}$ by the global sign δ. $\square$

Lemma 3.19. *The nontrivial scalar coset in $\mathrm{Bi}_K(2) \setminus P\Gamma_K(2)$ fails the global sign condition.*

Proof. We evaluate the index $[\mathrm{Bi}_K(2) : P\Gamma_K(2)]$. An element $[M] \in \mathrm{Bi}_K(2)$ admits a representative $M \equiv \lambda I_2 \pmod{2\mathcal{O}_K}$ for some local unit $\lambda \in (\mathcal{O}_K/2\mathcal{O}_K)^\times$.

- *For $K = \mathbb{Q}(i)$:* The local unit group is $(\mathcal{O}_K/2\mathcal{O}_K)^\times = \{1, i\}$. The reduction map from the global units $\mathcal{O}_K^\times = \{\pm 1, \pm i\}$ to $\{1, i\}$ is surjective. We can choose a global unit $u \equiv \lambda \pmod 2$. The representative $u^{-1}M$ satisfies $u^{-1}M \equiv I_2 \pmod 2$, meaning $[M] \in P\Gamma_K(2)$. Hence, $\mathrm{Bi}_K(2) = P\Gamma_K(2)$.
- *For $K = \mathbb{Q}(\sqrt{-2})$:* The quotient ring is $\mathbb{F}_2[\tau]/(\tau^2)$ with unit group $\{1, \tau + 1\}$. Global units $\{\pm 1\}$ both reduce to 1, yielding $[\mathrm{Bi}_K(2) : P\Gamma_K(2)] \leq 2$.
- *For $K = \mathbb{Q}(\sqrt{-p})$:* Since $\tau^2 = -p \equiv 1 \pmod 2$, the quotient ring is $\mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2[\epsilon]/(\epsilon^2)$ where $\epsilon = \tau + 1$. The local unit group is $\{1, \tau\}$. Global units map only to 1, again yielding $[\mathrm{Bi}_K(2) : P\Gamma_K(2)] \leq 2$.

Since $\mathrm{Aut}(X_{K,2})$ is a subgroup containing $P\Gamma_K(2)$, and the index is at most 2, it suffices to test a single representative of the nontrivial coset $\mathrm{Bi}_K(2) \setminus P\Gamma_K(2)$ for the latter two fields.

- *For $K = \mathbb{Q}(\sqrt{-2})$:* Let $M_0 = \left(\begin{smallmatrix} 1+\tau & 2 \\ 2 & 1-\tau \end{smallmatrix}\right)$. We have $\det(M_0) = -1$. Because $M_0 \equiv (1+\tau)I_2 \pmod{2\mathcal{O}_K}$, it reduces to the nontrivial scalar, confirming the index is exactly 2. Evaluating its action on $P(1)/4$ modulo $4S_K$ yields:
$$M_0 P(1) M_0^* = \begin{pmatrix} 4 & 3+2\tau \\ 3-2\tau & 4 \end{pmatrix}.$$
The off-diagonal entry requires $(3+2\tau) \mp 1 \in 4\mathcal{O}_K$. However, $2+2\tau \notin 4\mathcal{O}_K$ and $4+2\tau \notin 4\mathcal{O}_K$. Thus, M_0 fails the global sign condition.
- *For $K = \mathbb{Q}(\sqrt{-p})$:* Let $s = (p-1)/4$ and $M_0 = \left(\begin{smallmatrix} \tau & 2 \\ -2s & \tau \end{smallmatrix}\right)$. We have $\det(M_0) = -1$. Again, M_0 reduces to the nontrivial scalar $\tau I_2 \pmod{2\mathcal{O}_K}$, confirming the index is exactly 2. Calculation gives $M_0 P(1) M_0^* = P(1)$, which requires the global sign to be $+1$. However, testing $P(\tau)$ modulo $4pS_K$ gives:
$$M_0 P(\tau) M_0^* = \begin{pmatrix} -4p & (2p-1)\tau \\ -(2p-1)\tau & -p(p-1) \end{pmatrix}.$$
This requires $(2p-1)\tau - \tau = 2(p-1)\tau \in 4p\mathcal{O}_K$, which is false.

□

Lemma 3.20. *No element of the extended ideal-class coset $\widehat{\mathrm{Bi}}_K \setminus \mathrm{Bi}_K$ acts on $A_{S_K(2)}$ by a global sign. Furthermore, the same calculation excludes the extended Galois coset $(\widehat{\mathrm{Bi}}_K \setminus \mathrm{Bi}_K)\sigma$.*

Proof. For $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-2})$, the class number is 1 and the discriminant yields no nontrivial 2-torsion ideal classes, so $\widehat{\mathrm{Bi}}_K = \mathrm{Bi}_K$. Using the description of Example 3.4. Computing its action on $P(1) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ directly:
$$W_2 P(1) W_2^* = \begin{pmatrix} 1+\tau & 2 \\ 2s & 1-\tau \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-\tau & 2s \\ 2 & 1+\tau \end{pmatrix} = \begin{pmatrix} 4 & 2\tau \\ -2\tau & 4s \end{pmatrix}.$$
Dividing by 2 and reducing modulo $2\mathcal{O}_K$, we obtain:
$$\frac{1}{2} W_2 P(1) W_2^* = \begin{pmatrix} 2 & \tau \\ -\tau & 2s \end{pmatrix} \equiv \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} = P(\tau) \pmod{2\mathcal{O}_K}.$$
Suppose an element of this extended coset is induced by an automorphism of $X_{K,2}$. It must have the form MW_2 for some ordinary Bianchi matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_K)$. By Proposition 3.8, MW_2 must act on $A_{S_K(2)}$ by a global sign ± 1. Because $+1 \equiv -1 \pmod 2$, the induced action of MW_2 on the level-2 subgroup $S_K/2S_K$ must be the identity. Its action on $P(1)$ must yield $P(1)$ modulo $2\mathcal{O}_K$. Composing the actions, we require:
$$M \left(\frac{1}{2} W_2 P(1) W_2^* \right) M^* \equiv P(1) \pmod{2\mathcal{O}_K},$$
which simplifies to $MP(\tau)M^* \equiv P(1) \pmod{2\mathcal{O}_K}$. We compute the left-hand side directly:
$$MP(\tau)M^* \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = \begin{pmatrix} \tau(a\overline{b} + b\overline{a}) & \tau(a\overline{d} + b\overline{c}) \\ \tau(c\overline{b} + d\overline{a}) & \tau(c\overline{d} + d\overline{c}) \end{pmatrix} \pmod{2\mathcal{O}_K}.$$
Since $2 \equiv 0 \pmod{2\mathcal{O}_K}$, the diagonal entries vanish. For the off-diagonal entry, complex conjugation is trivial modulo 2, so $a\overline{d} + b\overline{c} \equiv ad - bc = \det(M) \pmod{2\mathcal{O}_K}$. Thus, that congruence requires:
$$\begin{pmatrix} 0 & \tau\det(M) \\ \tau\det(M) & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2\mathcal{O}_K},$$

which implies $\tau \det(M) \equiv 1 \pmod{2\mathcal{O}_K}$. Since $\det(M) \in \{\pm 1\}$ and both signs reduce to 1 modulo 2, this forces $\tau \equiv 1 \pmod{2\mathcal{O}_K}$. However, this requires $\tau - 1 \in 2\mathcal{O}_K$. Equivalently, in the quotient ring $\mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2[\epsilon]/(\epsilon^2)$, the class $\epsilon = \tau + 1 = \tau - 1$ must be zero, which is false. Thus, no such matrix M can exist. Finally, since $\sigma(P(1)) = P(1)$, we have $\frac{1}{2}W_2\sigma(P(1))W_2^* = \frac{1}{2}W_2P(1)W_2^* \equiv P(\tau) \pmod{2\mathcal{O}_K}$. Therefore the same contradiction $\tau \det(M) \equiv 1 \pmod{2\mathcal{O}_K}$ follows, excluding the extended Galois coset as well. □

Proof of Theorem 3.17. By Lemma 2.6, the representation ρ is injective because the discriminant group $A_{S_K(2)}$ is not 2-elementary in all cases considered (it contains 4-torsion for $K = \mathbb{Q}(i)$, 8-torsion for $K = \mathbb{Q}(\sqrt{-2})$, and $4p$-torsion for $K = \mathbb{Q}(\sqrt{-p})$). By the Torelli theorem, since the lattice $S_K(2)$ contains no (-2)-classes, the set of roots is empty ($\Delta(S_K(2)) = \emptyset$) and the Weyl group is trivial ($W(S_K(2)) = 1$). Consequently, the ample cone coincides with the positive cone.

We have decomposed the base isometries into the ordinary Bianchi group, the extended ideal-class coset, and the Galois cosets. Lemma 3.10 rules out the ordinary Galois coset, and Lemma 3.20 rules out both the extended ideal-class coset and the extended Galois coset. We are therefore restricted to the ordinary Bianchi group Bi_K. By Proposition 3.8, the global sign condition forces any such element to belong to $\mathrm{Bi}_K(2)$. By Lemmas 3.18 and 3.19, the elements of $\mathrm{Bi}_K(2)$ satisfying the global sign condition are precisely those in the principal congruence subgroup $P\Gamma_K(2)$. Thus, $\rho(\mathrm{Aut}(X_{K,2})) = P\Gamma_K(2)$. Since ρ is injective, this gives $\mathrm{Aut}(X_{K,2}) \cong P\Gamma_K(2)$. □

We now turn to the study of higher-level congruence subgroups. The coincidence $\mathrm{Aut}(X_{K,2}) \cong P\Gamma_K(2)$ observed at level 2 might naturally suggest that the principal congruence subgroup $P\Gamma_K(2N)$ rather than the Bianchi congruence subgroup $\mathrm{Bi}_K(2N)$ is always the appropriate group to describe these automorphisms. We conclude this section with examples demonstrating that this assumption is false: at higher levels, the automorphism group is indeed isomorphic to a congruence subgroup, but not the principal one.

Theorem 3.21. *Let p be a prime such that $p \equiv 7 \pmod 8$, and let $K = \mathbb{Q}(\sqrt{-p})$. Let $N = q^e$, where $q \equiv 3 \pmod 4$ is an odd prime that splits in K. Then the automorphism group of level $2N$ coincides with the principal congruence subgroup of level $2N$:*

$$\mathrm{Aut}(X_{K,2N}) = P\Gamma_K(2N).$$

Proof. Since 2 splits in K, we have $\mathcal{O}_K^\times = \{\pm 1\}$. Let $g \in \mathrm{Aut}(X_{K,2N})$. Because $\mathrm{Aut}(X_{K,2N}) \subseteq \mathrm{Bi}_K(2N)$, we can choose a representative $M \in \mathrm{GL}_2(\mathcal{O}_K)$ such that $M \equiv uI_2 \pmod{2N\mathcal{O}_K}$ for some $u \in (\mathcal{O}_K/2N\mathcal{O}_K)^\times$. Let $\delta := \det(M) \in \{\pm 1\}$. Because q splits in K, there is an isomorphism $\mathcal{O}_K/q^e\mathcal{O}_K \cong \mathbb{Z}/q^e\mathbb{Z} \times \mathbb{Z}/q^e\mathbb{Z}$, under which complex conjugation exchanges the factors. Writing the q^e-component of u as $u_q = (a, b)$, the relation $\det(M) = \delta$ modulo $q^e\mathcal{O}_K$ implies $a^2 = b^2 = \delta$ in $\mathbb{Z}/q^e\mathbb{Z}$.

The action of g on the q^e-primary level component of the discriminant group is the norm scalar $\alpha_{q^e}(g) = u_q\overline{u_q} = (a, b)(b, a) = (ab, ab)$, which we identify with $ab \in (\mathbb{Z}/q^e\mathbb{Z})^\times$. At the same time, g acts on the original p-primary part via multiplication by $\epsilon_p(g) = \delta$. Since $g \in \mathrm{Aut}(X_{K,2N})$, all primary components share a common sign, meaning $ab = \delta$. Combining this with $a^2 = \delta$ yields $ab = a^2$, and since a is a unit, $a = b$. Thus, $u_q = (a, a)$ and $a^2 = \delta$.

If $\delta = -1$, then $a^2 \equiv -1 \pmod q$. However, $q \equiv 3 \pmod 4$ makes -1 a quadratic non-residue modulo q, so this is impossible. Therefore, $\delta = 1$. Consequently, $a^2 \equiv 1 \pmod{q^e}$, which yields $a \equiv \pm 1 \pmod{q^e}$ since q is odd. Thus, $M \equiv \varepsilon I_2 \pmod{q^e\mathcal{O}_K}$ for some $\varepsilon \in \{\pm 1\}$.

Modulo 2, $\mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2 \times \mathbb{F}_2$ has a trivial unit group, forcing $M \equiv I_2 \pmod{2\mathcal{O}_K}$ and $\varepsilon \equiv 1 \pmod{2\mathcal{O}_K}$. Since $2\mathcal{O}_K$ and $q^e\mathcal{O}_K$ are relatively prime, the Chinese remainder theorem gives $\varepsilon^{-1}M \equiv I_2 \pmod{2q^e\mathcal{O}_K}$. Hence $g \in P\Gamma_K(2N)$, proving $\mathrm{Aut}(X_{K,2N}) \subseteq P\Gamma_K(2N)$.

Conversely, let $g \in P\Gamma_K(2N)$. It has a representative $M \equiv I_2 \pmod{2N\mathcal{O}_K}$. The norm scalar on the q^e-primary level component is $\alpha_{q^e}(g) = 1$, and $\det(M) \equiv 1 \pmod{q\mathcal{O}_K}$. Since $\det(M) \in \{\pm 1\}$ and q is odd, $\det(M) = 1$, making $\epsilon_p(g) = 1$. Every primary component is acted on by $+1$, so $g \in \mathcal{C}_N$. This proves $P\Gamma_K(2N) \subseteq \mathrm{Aut}(X_{K,2N})$. □

Theorem 3.22. *Let $K = \mathbb{Q}(\sqrt{-p})$, where $p > 3$ is a prime satisfying $p \equiv 3 \pmod 4$. Let $N > 1$ be odd, and assume that every prime divisor $\ell \mid N$ is inert in K and congruent to 1 modulo 4.*

Let $X_{K,2N}$ be a very general $S_K(2N)$-polarized $K3$ surface. Then, under the normalized Bianchi action, the following hold:

(1) $\mathrm{Aut}(X_{K,2N}) \cong \mathrm{Bi}_K(2N)$.
(2) $P\Gamma_K(2N) \subsetneq \mathrm{Bi}_K(2N)$.

Proof. Proof of (1). Since $p \equiv 3 \pmod 4$, the fundamental discriminant of K is $D = -p$. In particular, D is odd. Moreover, $\mathcal{O}_K^\times = \{\pm 1\}$, because $p > 3$. Let $g = [M] \in \mathrm{Bi}_K(2N)$. Since $\det(M)$ is a unit, we may write $\delta := \det(M) \in \{\pm 1\}$. This sign is independent of the chosen integral representative of the projective class g. Fix a prime power $\ell^e \| N$. Since $g \in \mathrm{Bi}_K(2N)$, there exists $u_\ell \in \left(\mathcal{O}_K/\ell^e\mathcal{O}_K\right)^\times$ such that $M \equiv u_\ell I_2 \pmod{\ell^e\mathcal{O}_K}$. Taking determinants gives $u_\ell^2 = \delta$ in $\mathcal{O}_K/\ell^e\mathcal{O}_K$. Conjugating this equality gives $\overline{u}_\ell^2 = \delta$. Therefore $\left(\frac{\overline{u}_\ell}{u_\ell}\right)^2 = 1$. Since ℓ is odd and $\mathcal{O}_K/\ell^e\mathcal{O}_K$ is a local ring, it follows that $\frac{\overline{u}_\ell}{u_\ell} \in \{\pm 1\}$.

We determine this sign after reducing modulo ℓ. Since $\left(\frac{-p}{\ell}\right) = -1$, the prime ℓ is inert in K, and hence $\mathcal{O}_K/\ell\mathcal{O}_K \cong \mathbb{F}_{\ell^2}$. Conjugation on this residue field is the Frobenius automorphism.

(1) If $\delta = 1$, then u_ℓ is a root of $x^2 - 1$, so $u_\ell \equiv \pm 1 \pmod \ell$. Thus u_ℓ is fixed by conjugation modulo ℓ. Since u_ℓ and $\overline{u}_\ell$ are roots of the same polynomial and $2u_\ell$ is a unit, the lift is unique, and therefore $\overline{u}_\ell = u_\ell$ modulo ℓ^e.
(2) If $\delta = -1$. Then u_ℓ is a root of $x^2 + 1$. Since $\ell \equiv 1 \pmod 4$, the roots of $x^2 + 1$ belong to the prime field $\mathbb{F}_\ell$. They are therefore fixed by Frobenius. Again, uniqueness of the Hensel lift gives $\overline{u}_\ell = u_\ell$ modulo ℓ^e.

Thus, in both cases, $\alpha_{\ell^e}(g) = u_\ell\overline{u}_\ell = u_\ell^2 = \delta$. The only odd ramified prime of K is p. By the ramified prime calculation, the corresponding discriminant character is $\epsilon_p(g) = \det(M) = \delta$. Since D is odd, the 2-primary condition is automatic. Therefore all local discriminant characters agree with the single global sign $\sigma = \delta$. By the global-sign criterion, $g|_{A_{S_K(2N)}} = \delta\, \mathrm{id}_{A_{S_K(2N)}}$. Hence every element of $\mathrm{Bi}_K(2N)$ satisfies the criterion of Proposition 3.8.

Proof of (2). Set $m := 2N$. For every prime power $\ell^e \| N$, the congruence $x^2 \equiv -1 \pmod{\ell^e}$ has a solution because $\ell \equiv 1 \pmod 4$. Since $1^2 \equiv -1 \pmod 2$, the Chinese remainder theorem gives an integer u such that $u^2 \equiv -1 \pmod m$. Since $m > 2$, one has $u \not\equiv \pm 1 \pmod m$.

Write $s := \frac{u^2+1}{m} \in \mathbb{Z}$. Since u is invertible modulo m, choose $t \in \mathbb{Z}$ such that $ut \equiv -s \pmod m$. Then $c := \frac{s+ut}{m}$ is an integer. Define $M = \begin{pmatrix} u & m \\ mc & u+mt \end{pmatrix}$. A direct computation gives $\det(M) = -1$. Thus $M \in \mathrm{GL}_2(\mathbb{Z}) \subseteq \mathrm{GL}_2(\mathcal{O}_K)$. Moreover, $M \equiv uI_2 \pmod{m\mathcal{O}_K}$. Since u is invertible modulo m, the projective reduction of M is trivial, and therefore $[M] \in \mathrm{Bi}_K(2N)$.

Suppose that $[M] \in P\Gamma_K(2N)$. Then some global unit $\varepsilon \in \mathcal{O}_K^\times = \{\pm 1\}$ satisfies $\varepsilon M \equiv I_2 \pmod{m\mathcal{O}_K}$. Since $M \equiv uI_2$, this implies $\varepsilon u \equiv 1 \pmod m$, and hence $u \equiv \pm 1 \pmod m$. This contradicts the choice of u. Therefore $[M] \in \mathrm{Bi}_K(2N) \setminus P\Gamma_K(2N)$, and consequently $P\Gamma_K(2N) \subsetneq \mathrm{Bi}_K(2N)$. Combining the two parts proves $P\Gamma_K(2N) \subsetneq \mathrm{Bi}_K(2N) \cong \mathrm{Aut}(X_{K,2N})$. □

3.3. **Mordell–Weil group.** We now examine the basic geometry of these $K3$ surfaces. These observations help us construct concrete models of $X_{K,2}$. From the structure of the lattice $S_K(2N)$, we can immediately deduce the following properties.

Proposition 3.23. *Let $X_{K,2N}$ be a $K3$ surface with $\mathrm{NS}(X_{K,2N}) \cong S_K(2N)$. Then:*

(1) *Each of the primitive isotropic classes e_1, e_2 defines a genus-one fibration.*
(2) *$S_K(2N)$ does not represent -2; hence $X_{K,2N}$ contains no smooth rational curves.*

To provide a useful contrast, we first recall Totaro's non-arithmeticity criterion for $K3$ surfaces:

Theorem 3.24. [15, Corollary 6.2] *Let X be a $K3$ surface with Picard number $\rho(X) \geq 4$. Suppose that (1) X admits a genus-one fibration with no reducible fibers, (2) it admits a second genus-one fibration with positive Mordell–Weil rank, and (3) X contains a (-2)-curve. Then the automorphism group $\mathrm{Aut}(X) \subset O(\mathrm{Pic}(X))$ is not commensurable with an arithmetic group.*

Our $K3$ surfaces $X_{K,2N}$ contrast with Totaro's situation: they satisfy conditions (1) and (2) but fail condition (3). The absence of smooth rational curves implies that the Weyl group is trivial ($W = 1$).

In the next section, we will review the basics of the Mordell–Weil group of the Jacobian fibration.

Proposition 3.25. *Let $X = X_{K,2N}$ be a very general member of the family $\mathrm{NS}(X) \cong S_K(2N)$. Let $F \in \mathrm{NS}(X)$ be the primitive nef isotropic class corresponding to a cusp of the Bianchi model, and let*

$$\pi_F : X \to \mathbb{P}^1$$

be the induced genus-one fibration. Let $J_F \to \mathbb{P}^1$ be its associated Jacobian fibration. Then the unipotent cusp subgroup of $\mathrm{Aut}(X)$ stabilizing F is naturally identified with the Mordell–Weil group of J_F: $U_F \cong \mathrm{MW}(J_F)$. Under the Bianchi identification of the cusp stabilizer at F, one has

$$U_F \cong \left\{ \left[\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \right] : \lambda \in 2N\mathcal{O}_K \right\} \cong 2N\mathcal{O}_K \cong \mathbb{Z}^2.$$

Proof. Since $F^2 = 0$ and $F \neq 0$, the Hodge index theorem implies that $F^\perp/\mathbb{Z}F$ is negative definite. Hence $O(F^\perp/\mathbb{Z}F)$ is finite. Therefore any unipotent element in the stabilizer of F acts trivially on $F^\perp/\mathbb{Z}F$.

We use the following standard facts about Mordell–Weil groups of elliptic surfaces. For a Jacobian fibration $\phi : J \to \mathbb{P}^1$ with zero section, every section acts on the generic fiber by translation, and, since a $K3$ surface is minimal, the resulting birational self-map extends to an automorphism of J; see [14, Section 4, equation (4.1)]. Equivalently, there is a natural embedding $\mathrm{MW}(\phi) \hookrightarrow \mathrm{Aut}(J)$. For a genus-one fibration $\pi_F : X \to \mathbb{P}^1$, its associated Jacobian fibration $J_F \to \mathbb{P}^1$ acts fiberwise on X through the usual torsor action of the relative Jacobian. Thus the fiberwise translation subgroup of π_F is identified with $\mathrm{MW}(J_F)$.

We now explain why this group is torsion-free in our situation. Let $\phi : J_F \to \mathbb{P}^1$ be the associated Jacobian fibration, let f be the fiber class, and let z be the zero section. The classes f, z span the hyperbolic plane $U_\phi = \langle f, z \rangle \subset \mathrm{NS}(J_F)$. Let Σ_ϕ denote the root lattice generated by the components of reducible fibers disjoint from the zero section. By [14, Proposition 4.1], Σ_ϕ is precisely the lattice generated by those fiber components, and by [14, Theorem 4.3] one has an isomorphism

$$\mathrm{MW}(J_F) \cong \mathrm{NS}(J_F)/(U_\phi \oplus \Sigma_\phi).$$

In our case X has no (-2)-curves. Hence the genus-one fibration π_F has no reducible fibers; equivalently the associated Jacobian fibration has no reducible-fiber root lattice, so $\Sigma_\phi = 0$. Therefore

$$\mathrm{MW}(J_F) \cong \mathrm{NS}(J_F)/U_\phi.$$

Since $U_\phi \cong U$ is unimodular, it is a primitive direct summand of $\mathrm{NS}(J_F)$. Hence the quotient $\mathrm{NS}(J_F)/U_\phi$ is torsion-free. Consequently, $\mathrm{MW}(J_F) = \mathrm{MW}(J_F)_{\mathrm{free}}$. Combining this with the lattice-theoretic description of the unipotent cusp stabilizer gives

$$U_F \cong \mathrm{MW}(J_F).$$

Finally, in the Bianchi model, the cusp stabilizer corresponding to F is the level-$2N$ parabolic subgroup. Its unipotent radical is

$$\lambda \longmapsto \left[\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right], \qquad \lambda \in 2N\mathcal{O}_K.$$

Since $\mathcal{O}_K$ is a rank-two $\mathbb{Z}$-lattice, this group is isomorphic to $\mathbb{Z}^2$. □

4. Deformations of Kummer surfaces

In this section, we construct the level-two $K3$ surfaces $X_{K,2}$ as very general deformations of Kummer surfaces. The main point is that the non-exceptional lattice $S_K(2)$ is primitive in the orthogonal complement of the Kummer lattice. This allows us to deform away the exceptional Kummer classes while keeping precisely the rank-four lattice $S_K(2)$ algebraic.

4.1. **Kummer surfaces.** We first recall the classical construction of Kummer surfaces. Let A be an abelian surface and let $\iota : A \longrightarrow A,\ x \longmapsto -x$ be the natural involution.

Construction 4.1. Let $G = \{1, \iota\} \cong \mathbb{Z}/2\mathbb{Z}$ act on A, and set $Y = A/G$. The fixed locus of ι is the group $A[2]$ of two-torsion points, which has cardinality 16. Hence Y has exactly 16 ordinary double points, one for each point of $A[2]$. Away from these singularities the quotient is étale in codimension one, and the canonical divisor satisfies $K_Y \sim 0$.

Let $\pi : X \longrightarrow Y$ be the minimal resolution. Since the singularities are of type A_1, the resolution is crepant, and therefore $K_X = \pi^* K_Y \sim 0$. The exceptional locus consists of 16 disjoint smooth rational curves $E_1, \dots, E_{16}$ with $E_i^2 = -2$. The resulting smooth surface X is called the *Kummer surface* associated to A, and is denoted $X = \mathrm{Km}(A)$. It is a $K3$ surface.

Definition 4.2. Let $X = \mathrm{Km}(A)$. Let $E = \langle E_1, \dots, E_{16}\rangle \cong \langle -2\rangle^{\oplus 16}$ be the lattice generated by the sixteen exceptional curves. Its saturation in $H^2(X, \mathbb{Z})$ is called the *Kummer lattice* and will be denoted by $\mathcal{K}$. Thus $\mathcal{K} = (E \otimes \mathbb{Q}) \cap H^2(X, \mathbb{Z})$.

The Kummer lattice accounts for the exceptional part of the Picard group. The part coming from the abelian surface itself lies in the orthogonal complement of $\mathcal{K}$. The following standard fact is the basic reason for the appearance of a factor of 2 in Kummer constructions.

Lemma 4.3. *Let $X = \mathrm{Km}(A)$ and let $\mathcal{K} \subset H^2(X, \mathbb{Z})$ be the Kummer lattice. Then there is a Hodge isometry $\mathcal{K}^\perp \cong H^2(A, \mathbb{Z})(2)$. Under this identification, $\mathrm{NS}(X) \cap \mathcal{K}^\perp \cong \mathrm{NS}(A)(2)$. Then $\mathrm{NS}(A)(2)$ is primitive in $\mathcal{K}^\perp$.*

We now prove the existence of the required Kummer deformation.

Proposition 4.4. *Let X be a complex $K3$ surface, and let $S \subset \mathrm{NS}(X)$ be a primitive hyperbolic sublattice. Then there exists a deformation X' of X such that $\mathrm{Pic}(X') \cong S$. Moreover, for a very general such deformation, every class in $\Lambda_{K3} \setminus S$ is not of type $(1,1)$.*

Proof. Consider the period domain

$$\Omega_S = \left\{ [\omega] \in \mathbb{P}(S^\perp \otimes \mathbb{C}) \mid (\omega, \omega) = 0,\ (\omega, \overline{\omega}) > 0 \right\}.$$

For each integral class $\delta \in \Lambda_{K3} \setminus S$, the condition that δ remains algebraic is the hyperplane condition $(\omega, \delta) = 0$. Since there are only countably many such δ, a very general point of Ω_S avoids all these hyperplanes. By the surjectivity of the period map, such a period is realized by a marked $K3$ surface. By the Lefschetz $(1,1)$ theorem, its Picard lattice is exactly S. □

4.2. **Primitive lattices of abelian surfaces and the equational model.** Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. We fix an elliptic curve $E_K \coloneqq \mathbb{C}/\mathcal{O}_K$, and set $A_K \coloneqq E_K \times E_K$.

Definition 4.5. We define the standard geometric divisor classes on A_K:

(1) *Fiber divisors:* The horizontal and vertical fibers $F_1 = E_K \times \{0\}$ and $F_2 = \{0\} \times E_K$.
(2) *Graph divisors:* For $\alpha \in \mathcal{O}_K$, the graph of multiplication by α, denoted $\Gamma_\alpha = \{(x, \alpha x) \mid x \in E_K\} \subset A_K$.

Equivalently, in the Hermitian matrix model for $\mathrm{NS}(A_K)$, we use the classes $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $P(z) = \left(\begin{smallmatrix} 0 & z \\ \bar{z} & 0 \end{smallmatrix}\right)$. The classes e and f span a copy of U. If $\{1, \tau\}$ is an integral basis of $\mathcal{O}_K$, then the remaining two classes may be taken to be $P(1)$ and $P(\tau)$. Hence $\mathrm{NS}(A_K) \cong U \oplus M_K$, where M_K is the negative-definite CM trace lattice. We write $S_K := \mathrm{NS}(A_K) \cong U \oplus M_K$ and $S_K(2) := \mathrm{NS}(A_K)(2) \cong U(2) \oplus M_K(2)$.

Let $X_K^{\mathrm{Km}} = \mathrm{Km}(A_K)$. By Lemma 4.3, the non-exceptional part of the Picard lattice of X_K^{Km} is $\mathrm{NS}(X_K^{\mathrm{Km}}) \cap \mathcal{K}^\perp \cong \mathrm{NS}(A_K)(2) \cong S_K(2)$. In particular, $S_K(2)$ is primitive in $\mathrm{NS}(X_K^{\mathrm{Km}}) \cap \mathcal{K}^\perp$.

We now explain how this deformation is seen in the double-cover equation. Choose a Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ for E_K. The quotient map $E_K \longrightarrow E_K/\{\pm 1\} \cong \mathbb{P}^1$ is a double cover branched over the three finite roots of the cubic and over the point at infinity. Thus the correct homogeneous branch section is the binary quartic

$$B_K(X, Y) = (4X^3 - g_2 XY^2 - g_3 Y^3)Y \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)).$$

The singular Kummer quotient of $A_K = E_K \times E_K$ is therefore the double cover

$$Y_K^{\mathrm{sing}} = \left\{ S^2 = B_K(X, Y) B_K(U, V) \right\} \subset \mathrm{Spec}_{\mathbb{P}^1 \times \mathbb{P}^1} \mathrm{Sym}(\mathcal{O}(-2, -2)).$$

Its branch divisor is the union of four vertical and four horizontal fibers. Their 16 intersection points give the 16 ordinary double points of Y_K^{sing}, and resolving them gives X_K^{Km}.

A local equational deformation of this double cover is obtained by replacing the branch equation by a nearby $(4, 4)$-form:

$$S^2 = B_K(X, Y) B_K(U, V) + t H_{4,4}(X, Y; U, V),$$

where $H_{4,4} \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(4, 4))$. For a general choice of $H_{4,4}$, the branch curve is smooth and the double cover is a smooth $K3$ surface. The two ruling classes on $\mathbb{P}^1 \times \mathbb{P}^1$ pull back to classes e, f satisfying $e^2 = f^2 = 0$ and $e \cdot f = 2$, so this equational family naturally preserves a copy of $U(2)$.

However, a general $(4, 4)$-deformation does not preserve the graph classes. The additional graph classes are precisely $P(1)$ and $P(\tau)$. Geometrically, these come from the graphs of the endomorphisms $x \mapsto x$ and $x \mapsto \tau x$ on E_K. After passing to the Kummer quotient and resolving its singularities, their classes lie in $\mathrm{NS}(X_K^{\mathrm{Km}}) \cap \mathcal{K}^\perp \cong S_K(2)$.

Proposition 4.6. *There exists a nonempty Noether–Lefschetz locus in the local* $(4, 4)$*-double-cover family such that a very general member* X *of this locus satisfies*

$$\mathrm{Pic}(X) \cong S_K(2) = U(2) \oplus M_K(2).$$

Proof. Let $X_K^{\mathrm{Km}} = \mathrm{Km}(E_K \times E_K)$. By Lemma 4.3, $S_K(2)$ is a primitive sublattice of $K^\perp$, hence as a primitive sublattice of the marked $K3$ lattice. The two fiber classes on $E_K \times E_K$ give the $U(2)$-summand, while the graph classes associated with the endomorphisms 1 and τ give the two additional classes $P(1)$, $P(\tau)$.

Now consider the period domain

$$\Omega_{S_K(2)} = \left\{ [\omega] \in \mathbb{P}\big(S_K(2)^\perp \otimes \mathbb{C}\big) \;\middle|\; (\omega, \omega) = 0,\ (\omega, \overline{\omega}) > 0 \right\}.$$

For every integral class $\delta \in \Lambda_{K3} \setminus S_K(2)$, the condition that δ remains algebraic is the hyperplane condition $(\omega, \delta) = 0$. Since there are only countably many such classes δ, a very general period point of $\Omega_{S_K(2)}$ avoids all these hyperplanes. By the surjectivity of the period map, there exists a $K3$ surface X, arbitrarily close to X_K^{Km} in the $S_K(2)$-polarized deformation space, such that $\mathrm{Pic}(X) \cong S_K(2)$.

It remains to see that such a surface belongs to the $(4,4)$-double-cover family. Let $F_1, F_2 \in \mathrm{Pic}(X)$ be the two primitive isotropic classes spanning the $U(2)$-summand. Thus

$$F_1^2 = F_2^2 = 0, \qquad F_1 \cdot F_2 = 2.$$

Since $S_K(2)$ contains no classes of square -2, the nef cone of X coincides with the positive cone. After choosing the positive cone component, we may assume that F_1 and F_2 are nef.

By [6, Chapter 2 Proposition 3.10] the complete linear systems $|F_1|$ and $|F_2|$ define genus-one fibrations

$$\varphi_i : X \longrightarrow \mathbb{P}^1, \qquad i = 1, 2.$$

Consider the product morphism $\varphi = (\varphi_1, \varphi_2) : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Since $F_1 \cdot F_2 = 2$, the morphism φ is generically finite of degree 2. Moreover, Φ contracts no curve. Indeed, if an irreducible curve C were contracted, then $C \cdot F_1 = C \cdot F_2 = 0$. Hence $[C]$ would lie in the negative-definite orthogonal complement of $U(2)$ in $S_K(2)$. Since $S_K(2)$ contains no (-2)-classes, this is impossible for an irreducible curve on a $K3$ surface. Thus Φ is finite. Therefore X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$. Since $K_X \sim 0$, the canonical bundle formula for double covers shows that the branch divisor has class $-2K_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4)$. Thus X is realized as a smooth double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a smooth curve of bidegree $(4,4)$. Since the two nef classes F_1, F_2 deform the two ruling classes of the Kummer double cover, the resulting branch divisor is a small deformation of $B_K(X,Y)B_K(U,V) = 0$. Thus this construction lies in the local $(4,4)$-double-cover family considered above.

Under the $U(2)$-polarized period map for the $(4,4)$-double-cover family, the conditions that $P(1)$ and $P(\tau)$ remain algebraic are precisely the two Hodge equations

$$(\omega, P(1)) = 0, \qquad (\omega, P(\tau)) = 0.$$

Hence they cut out a local Noether–Lefschetz locus in the $(4,4)$-double-cover family. The preceding construction shows that this locus is nonempty. For a very general point of this locus, the period avoids all hyperplanes associated with classes outside $S_K(2)$, and therefore

$$\mathrm{Pic}(X) \cong S_K(2) = U(2) \oplus M_K(2).$$

□

$$\begin{tikzcd}
& Y_{K,2} \arrow[dl, squiggly] \arrow[d, "\mathrm{Bl}_{p_1,\ldots,p_{16}}"] \\
X_{K,2} & A_K/\langle \pm 1 \rangle \arrow[l, squiggly]
\end{tikzcd}$$

FIGURE 1. The Kummer surface $Y_{K,2}$ resolves the singular quotient $A_K/\langle \pm 1\rangle$, and both lie on the boundary of the deformation family whose very general member is $X_{K,2}$.

4.3. **Parabolic generation.** We now study the geometric generators of $\mathrm{Aut}(X_{K,2})$ in the cases $K = \mathbb{Q}(\sqrt{-2})$ and $K = \mathbb{Q}(\sqrt{-7})$. We begin by defining the relevant congruence subgroup and establishing how the projective group splits over the principal congruence subgroup.

Definition 4.7. Let $K = \mathbb{Q}(\sqrt{-d})$. We define

$$\Gamma_K(2) := \{[M] \in \mathrm{PSL}_2(\mathcal{O}_K) \mid \text{there exists } M \in \mathrm{SL}_2(\mathcal{O}_K) \text{ with } M \equiv I_2 \pmod{2\mathcal{O}_K}\}.$$

Lemma 4.8. *Let $K = \mathbb{Q}(\sqrt{-d})$ with $d \in \{2, 7\}$. Then $P\Gamma_K(2) = \Gamma_K(2) \rtimes \langle \eta \rangle$, where $\eta = [J]$ is represented by $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover, under the Hermitian action on $S_K = \mathrm{Herm}_2(\mathcal{O}_K)$, the element η acts by*

$$e \longmapsto e, \qquad f \longmapsto f, \qquad P(z) \longmapsto -P(z).$$

Proof. Since $d \in \{2, 7\}$, we have $\mathcal{O}_K^\times = \{\pm 1\}$. Hence the determinant gives a well-defined homomorphism $\det : P\Gamma_K(2) \longrightarrow \{\pm 1\}$, because replacing a representative M by $-M$ does not change $\det(M)$.

The kernel of this homomorphism is exactly $\Gamma_K(2)$. Indeed, if $[M] \in P\Gamma_K(2)$ and $\det(M) = 1$, then M may be chosen in $\mathrm{SL}_2(\mathcal{O}_K)$ with

$$M \equiv I_2 \pmod{2\mathcal{O}_K},$$

so $[M] \in \Gamma_K(2)$. Conversely, every element of $\Gamma_K(2)$ has determinant 1.

The element $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies $J \equiv I_2 \pmod{2\mathcal{O}_K}$ because $-1 \equiv 1 \pmod{2\mathcal{O}_K}$, and $\det(J) = -1$. Therefore $\eta = [J]$ gives a splitting of the determinant map, and we obtain

$$P\Gamma_K(2) = \Gamma_K(2) \rtimes \langle \eta \rangle.$$

Finally, for

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P(z) = \begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix},$$

a direct computation gives

$$JeJ^* = e, \qquad JfJ^* = f, \qquad JP(z)J^* = -P(z).$$

This proves the claim. □

One notable consequence of the decomposition $P\Gamma_K(2) = \Gamma_K(2) \rtimes \langle \eta \rangle$ is that the special-linear congruence subgroup contributes no torsion. Every torsion element of $P\Gamma_K(2)$ lies in the nontrivial coset $\Gamma_K(2)\eta$. Moreover, composing η with the parabolic translations in $\Gamma_K(2)$ produces infinitely many distinct involutions.

Lemma 4.9. *Let K be an imaginary quadratic field.*

(1) *The group $\Gamma_K(2)$ is torsion-free.*
(2) *The group $P\Gamma_K(2)$ contains infinitely many distinct involutions.*
(3) *Every nontrivial finite-order element of $\mathrm{Bi}_K(2)$, and hence of $P\Gamma_K(2)$, has order 2.*

Proof. We first prove that $\Gamma_K(2)$ is torsion-free. Let $[M] \in \Gamma_K(2)$ have finite order. By definition, we may choose a representative $M \in \mathrm{SL}_2(\mathcal{O}_K)$ satisfying $M \equiv I_2 \pmod{2\mathcal{O}_K}$. Write $M = I_2 + 2A$ with $A \in M_2(\mathcal{O}_K)$. Since $\det(M) = 1$, we have

$$1 = \det(I_2 + 2A) = 1 + 2\,\mathrm{tr}(A) + 4\det(A),$$

and therefore $\mathrm{tr}(M) = 2 + 2\,\mathrm{tr}(A) = 2 - 4\det(A)$.

Because $[M]$ has finite order in $\mathrm{PSL}_2(\mathcal{O}_K)$, some power of M is equal to $\pm I_2$. Hence the eigenvalues of M are roots of unity. Since $\det(M) = 1$, its trace is real, and thus $\mathrm{tr}(M) \in \mathcal{O}_K \cap \mathbb{R} = \mathbb{Z}$. Moreover, its eigenvalues have absolute value one, so $\mathrm{tr}(M) \in \{-2, -1, 0, 1, 2\}$. On the other hand,

$$\det(A) = \frac{2 - \mathrm{tr}(M)}{4} \in \mathcal{O}_K \cap \mathbb{Q} = \mathbb{Z}.$$

Consequently, $\mathrm{tr}(M) \equiv 2 \pmod 4$. Among the five possible traces above, this leaves only $\mathrm{tr}(M) = \pm 2$. A finite-order matrix over a field of characteristic zero is semisimple. Hence $M = I_2$ when $\mathrm{tr}(M) = 2$, and $M = -I_2$ when $\mathrm{tr}(M) = -2$. In either case, $[M]$ is the identity in $\mathrm{PSL}_2(\mathcal{O}_K)$. This proves (1).

For $z \in \mathcal{O}_K$, set

$$U_z := \begin{pmatrix} 1 & 2z \\ 0 & 1 \end{pmatrix} \in \Gamma_K(2), \qquad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$U_z J = \begin{pmatrix} 1 & -2z \\ 0 & -1 \end{pmatrix} \equiv I_2 \pmod{2\mathcal{O}_K},$$

and therefore $\tau_z := [U_z J] \in P\Gamma_K(2)$. Furthermore, $(U_z J)^2 = I_2$, so τ_z is an involution.

The elements τ_z are pairwise distinct. Indeed, if $\tau_z = \tau_w$ in $\mathrm{PGL}_2(\mathcal{O}_K)$, then

$$\begin{pmatrix} 1 & -2z \\ 0 & -1 \end{pmatrix} = u \begin{pmatrix} 1 & -2w \\ 0 & -1 \end{pmatrix}$$

for some $u \in \mathcal{O}_K^\times$. Comparing the $(1,1)$-entries gives $u = 1$, and then comparing the $(1,2)$-entries gives $z = w$. Since $\mathcal{O}_K$ is infinite, this proves (2).

We finally determine the possible orders of torsion elements in $\mathrm{Bi}_K(2)$. Let $[M] \in \mathrm{Bi}_K(2)$ have finite order, and choose a representative $M \in \mathrm{GL}_2(\mathcal{O}_K)$. Since $[M]$ lies in the projective level-two kernel, there exists $a \in \mathcal{O}_K$ such that $M \equiv aI_2 \pmod{2\mathcal{O}_K}$. Set $t := \mathrm{tr}(M)$ and $\delta := \det(M) \in \mathcal{O}_K^\times$. The preceding congruence gives $t \equiv 2a \equiv 0 \pmod{2\mathcal{O}_K}$, so $t \in 2\mathcal{O}_K$.

Let λ_1 and λ_2 be the eigenvalues of M. Since $[M]$ has finite order in the projective group, the ratio $\zeta := \frac{\lambda_1}{\lambda_2}$ is a root of unity. Consider the projectively invariant quantity $q := \frac{t^2}{\delta}$. Using $t = \lambda_1 + \lambda_2$ and $\delta = \lambda_1\lambda_2$, we obtain $q = 2 + \zeta + \zeta^{-1}$. In particular, q is real and $0 \le q \le 4$. Moreover, since δ is a unit, $q \in \mathcal{O}_K$. Thus $q \in \mathcal{O}_K \cap \mathbb{R} = \mathbb{Z}$. Since $t \in 2\mathcal{O}_K$, we also have $q = \frac{t^2}{\delta} \in 4\mathcal{O}_K$. Therefore $q \in 4\mathcal{O}_K \cap \mathbb{Z} = 4\mathbb{Z}$. Together with $0 \le q \le 4$, this gives $q \in \{0, 4\}$.

If $q = 4$, then $\zeta + \zeta^{-1} = 2$, and hence $\zeta = 1$. Since a projective finite-order matrix is semisimple, M is scalar, so $[M]$ is the identity.

Thus every nontrivial finite-order element satisfies $q = 0$. It follows that $t = 0$, and the Cayley–Hamilton identity gives $M^2 - tM + \delta I_2 = 0$, hence $M^2 = -\delta I_2$. Therefore $[M]^2 = 1$ in $\mathrm{PGL}_2(\mathcal{O}_K)$. This proves (3). ☐

The next point is to identify the covering involution in the Hermitian lattice model.

Lemma 4.10. *Let X be a very general member of the $S_K(2)$-polarized Noether–Lefschetz locus in the $(4,4)$-double-cover family $\pi : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Let $\iota : X \longrightarrow X$ be the covering involution. Under the identification*

$$\mathrm{Pic}(X) \cong S_K(2) = U(2) \oplus M_K(2),$$

where $U(2)$ is generated by the pullbacks of the two ruling classes and $M_K(2)$ is generated by the graph classes $P(1)$ and $P(\tau)$, the action of ι^* is given by

$$e \longmapsto e, \qquad f \longmapsto f, \qquad P(z) \longmapsto -P(z).$$

Equivalently, ι^* agrees with the Hermitian action of $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof. The quotient of X by the involution ι is $\mathbb{P}^1 \times \mathbb{P}^1$. Hence the invariant part of $H^2(X, \mathbb{Q})$ under ι^* is

$$H^2(X, \mathbb{Q})^{\iota^*} = \pi^* H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q}).$$

In particular, on $\mathrm{Pic}(X)$, the invariant sublattice is generated by the pullbacks of the two ruling classes. These are precisely the classes e and f, and they span the $U(2)$-summand.

Since X is very general in the $S_K(2)$-polarized Noether–Lefschetz locus, we have $\mathrm{Pic}(X) = U(2) \oplus M_K(2)$. Therefore the orthogonal complement of $U(2)$ in $\mathrm{Pic}(X)$ is exactly $M_K(2)$, generated by $P(1)$ and $P(\tau)$.

Because the ι^*-invariant part of $\mathrm{Pic}(X)_{\mathbb{Q}}$ is already $U(2)_{\mathbb{Q}}$, the involution ι^* acts as $-\mathrm{id}$ on $M_K(2)_{\mathbb{Q}}$. Since $M_K(2)$ is an integral lattice preserved by ι^*, this gives

$$\iota^* P(1) = -P(1), \qquad \iota^* P(\tau) = -P(\tau).$$

On the other hand, the Hermitian action of $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ sends

$$e \longmapsto e, \qquad f \longmapsto f, \qquad P(z) \longmapsto -P(z).$$

Thus ι^* agrees with the action of J on $S_K(2)$. □

We now establish that the congruence subgroup is generated by parabolic elements, count the relevant cusps, and map these parabolic stabilizers to the Mordell-Weil groups of the genus-one fibrations.

Theorem 4.11. *Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of class number one, and assume $d \notin \{1, 3\}$. Then the level-two principal congruence subgroup $\Gamma_K(2)$ is generated by its parabolic elements if and only if $d \in \{2, 7\}$.*

Proof. By the theorem of Baker, Goerner and Reid [1], among the class-number-one imaginary quadratic fields, the level-two principal congruence subgroup $\Gamma_K(2)$ is generated by parabolic elements precisely for $d \in \{1, 2, 3, 7\}$. Under the standing assumption $d \notin \{1, 3\}$, this list reduces exactly to $d \in \{2, 7\}$. This proves the claim. □

Proposition 4.12. *Let $K = \mathbb{Q}(\sqrt{-d})$ have class number one. Then the cusps of $\mathbb{H}^3/\Gamma_K(2)$ are naturally in bijection with $\mathbb{P}^1(R_2)$, where $R_2 = \mathcal{O}_K/2\mathcal{O}_K$. In particular, $\#\,\mathrm{Cusps}(\mathbb{H}^3/\Gamma_{\mathbb{Q}(\sqrt{-2})}(2)) = 6$, whereas $\#\,\mathrm{Cusps}(\mathbb{H}^3/\Gamma_{\mathbb{Q}(\sqrt{-7})}(2)) = 9$.*

Proof. Since K has class number one, the group $\mathrm{PSL}_2(\mathcal{O}_K)$ acts transitively on $\mathbb{P}^1(K) = K \cup \{\infty\}$. Passing to the principal congruence subgroup of level 2, the cusp orbits are obtained by reduction modulo 2. Therefore $\Gamma_K(2)\backslash\mathbb{P}^1(K) \cong \mathbb{P}^1(R_2)$.

We now count this finite projective line in the two cases. First let $K = \mathbb{Q}(\sqrt{-2})$. Then $R_2 = \mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2[t]/(t^2)$, where t is the image of $\sqrt{-2}$. This is a local ring with maximal ideal (t). Every point of $\mathbb{P}^1(R_2)$ is represented uniquely either in the form $[1 : a]$ for $a \in R_2$, or in the form $[b : 1]$ for $b \in (t)$. The first type gives $\#R_2 = 4$ points, and the second type gives $\#(t) = 2$ points. Hence $\#\mathbb{P}^1(R_2) = 4 + 2 = 6$.

Next let $K = \mathbb{Q}(\sqrt{-7})$. Since 2 splits in $\mathcal{O}_K$, we have $R_2 = \mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2 \times \mathbb{F}_2$. Therefore $\mathbb{P}^1(R_2) \cong \mathbb{P}^1(\mathbb{F}_2) \times \mathbb{P}^1(\mathbb{F}_2)$. Since $\#\mathbb{P}^1(\mathbb{F}_2) = 3$, we obtain $\#\mathbb{P}^1(R_2) = 3 \cdot 3 = 9$. This proves the claim. □

Lemma 4.13. *Let X be a very general $S_K(2)$-polarized K3 surface, and let $\ell_c \in S_K(2)$ be a primitive nef isotropic class. Let $\varphi_{\ell_c} : X \longrightarrow \mathbb{P}^1$ be the genus-one fibration defined by ℓ_c. Then the translation subgroup of the parabolic stabilizer of ℓ_c is naturally identified with the Mordell–Weil group $\mathrm{MW}(J_{\ell_c})$.*

Proof. The primitive nef isotropic class ℓ_c defines a genus-one fibration $\varphi_{\ell_c} : X \longrightarrow \mathbb{P}^1$. The stabilizer of the ray $\mathbb{R}_{\geq 0}\ell_c$ in the automorphism group is the automorphism group of this genus-one fibration. Its parabolic translation subgroup consists exactly of automorphisms acting fiberwise by translations. For a genus-one fibration on a $K3$ surface, the group of fiberwise translations is the Mordell–Weil group of the fibration. Hence this parabolic translation subgroup is identified with $\mathrm{MW}(J_{\ell_c})$. □

Finally, we synthesize the algebraic splitting, the geometric covering involution, and the parabolic generators into a complete description of the automorphism group.

Corollary 4.14. *Let $K = \mathbb{Q}(\sqrt{-d})$ with $d \in \{2,7\}$, and let $X_{K,2}$ be a very general $S_K(2)$-polarized double-cover deformation of $\mathrm{Km}(E_K \times E_K)$. Then $\mathrm{Aut}(X_{K,2}) \cong \Gamma_K(2) \rtimes \langle \iota \rangle$, where ι is the covering involution. Moreover, the subgroup $\Gamma_K(2)$ is generated by the Mordell–Weil translation groups associated to genus-one fibrations: $\Gamma_K(2) = \langle \mathrm{MW}(J_{\ell_c}) \mid c \in \Gamma_K(2)\backslash\mathbb{P}^1(K)\rangle$.*

For $K = \mathbb{Q}(\sqrt{-2})$, there are 6 cusp orbits. The standard cusp translation subgroup is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2\sqrt{-2} \\ 0 & 1 \end{pmatrix}$. For $K = \mathbb{Q}(\sqrt{-7})$, there are 9 cusp orbits. The standard cusp translation subgroup is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1+\sqrt{-7} \\ 0 & 1 \end{pmatrix}$.

Proof. By Lemma 4.8, we have $P\Gamma_K(2) = \Gamma_K(2) \rtimes \langle [J] \rangle$, and $[J]$ is realized geometrically by the covering involution ι. Using the Torelli identification $\mathrm{Aut}(X_{K,2}) \cong P\Gamma_K(2)$, we therefore obtain $\mathrm{Aut}(X_{K,2}) \cong \Gamma_K(2) \rtimes \langle \iota \rangle$.

By Theorem 4.11, the group $\Gamma_K(2)$ is generated by its parabolic elements for $d \in \{2,7\}$. Each cusp corresponds to a primitive isotropic ray in the positive cone, hence to a genus-one fibration on $X_{K,2}$. By Lemma 4.13, the parabolic translation subgroup at this cusp is the Mordell–Weil group of the corresponding genus-one fibration. Therefore $\Gamma_K(2) = \langle \mathrm{MW}(J_{\ell_c}) \mid c \in \Gamma_K(2)\backslash\mathbb{P}^1(K)\rangle$.

Finally, the number of cusp orbits is computed in Proposition 4.12. It is 6 for $K = \mathbb{Q}(\sqrt{-2})$ and 9 for $K = \mathbb{Q}(\sqrt{-7})$. The displayed matrices are the standard translation generators at the cusp ∞, corresponding respectively to the rank-two lattices $2\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ and $2\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$. ☐

5. Complete-intersection models

We now treat the two fields $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-3})$. In these cases the surfaces $X_{K,2}$ admit especially symmetric complete-intersection models, and the corresponding automorphism groups are generated by natural deck involutions.

Definition 5.1. Let $\mathbb{P}_I := \mathbb{P}^5 \times (\mathbb{P}^1)^3$, and let h, e_1, e_2, e_3 denote the hyperplane classes pulled back from the four factors. A $K3$ surface of *type I* is a smooth complete intersection

$$X_I = D_{11} \cap D_{12} \cap D_{21} \cap D_{22} \cap D_{31} \cap D_{32} \subset \mathbb{P}_I,$$

where

$$D_{i1}, D_{i2} \in |h + e_i|, \qquad i = 1,2,3.$$

Let $\mathbb{P}_{II} := (\mathbb{P}^1)^4$, and let g_1, g_2, g_3, g_4 denote the hyperplane classes pulled back from the four factors. A $K3$ surface of *type II* is a smooth complete intersection

$$X_{II} = D_1 \cap D_2 \subset \mathbb{P}_{II}, \qquad D_1, D_2 \in |g_1 + g_2 + g_3 + g_4|.$$

By the adjunction formula, these are indeed $K3$ surfaces.

Proposition 5.2. *Let X_I be a type I surface, and set*

$$H := h|_{X_I}, \qquad E_i := e_i|_{X_I} \quad (i = 1,2,3).$$

The lattice $G_I = \langle H, E_1, E_2, E_3 \rangle$ generated by these restrictions has the Gram matrix

$$G_I = \begin{pmatrix} 8 & 4 & 4 & 4 \\ 4 & 0 & 2 & 2 \\ 4 & 2 & 0 & 2 \\ 4 & 2 & 2 & 0 \end{pmatrix}.$$

Moreover, $G_I \cong S_{\mathbb{Q}(i)}(2)$, and $S_{\mathbb{Q}(i)}(2)$ is primitive in $\mathrm{Pic}(X_I)$.

Proof. In the Chow ring of $\mathbb{P}^5 \times (\mathbb{P}^1)^3$, we have $h^6 = 0$ and $e_i^2 = 0$. The class of X_I is $[X_I] = (h+e_1)^2(h+e_2)^2(h+e_3)^2$. Intersections on X_I are computed by taking the coefficient of $h^5 e_1 e_2 e_3$. This yields $H^2 = 8$, $H \cdot E_i = 4$, $E_i^2 = 0$, and $E_i \cdot E_j = 2$ (for $i \neq j$), resulting in the displayed Gram matrix. With respect to the unimodular change of basis E_1, E_2, $E_1 + E_2 - E_3$, $H - E_1 - E_2 - E_3$, the Gram matrix becomes $U(2) \perp \langle -4 \rangle \perp \langle -4 \rangle$.

To show primitivity, assume $S_{\mathbb{Q}(i)}(2)$ is not primitive. Then there exists an element $v \in S_{\mathbb{Q}(i)}^\vee(2) \setminus S_{\mathbb{Q}(i)}(2)$ such that $v \in \mathrm{Pic}(X_I)$ and $v^2 \in 2\mathbb{Z}$. A direct computation in the discriminant group A_{G_I} shows that the only nonzero classes which can generate an even overlattice are, up to permutation of E_1, E_2, E_3, represented by $\frac{1}{2}E_1$, $\frac{1}{2}(H+E_1)$, $\frac{1}{2}H$. Each leads to a geometric contradiction:

(1) $v \equiv \frac{1}{2}E_1 \pmod{S_{\mathbb{Q}(i)}(2)}$. Set $w := \frac{1}{2}E_1 - E_2$. We have $w^2 = -2$, so by Riemann-Roch, either w or $-w$ is effective. However, $w \cdot E_3 = -1$ and $(-w) \cdot E_2 = -1$, contradicting the nefness of the base-point free divisors E_2 and E_3.
(2) $v \equiv \frac{1}{2}(H+E_1) \pmod{S_{\mathbb{Q}(i)}(2)}$. Set $w := \frac{1}{2}E_1 - \frac{1}{2}H + E_2$. Here $w^2 = -2$, implying w or $-w$ is effective. However, $w \cdot E_2 = -1$ and $(-w) \cdot E_3 = -1$, again contradicting nefness.
(3) $v \equiv \frac{1}{2}H \pmod{S_{\mathbb{Q}(i)}(2)}$. Set $w := \frac{1}{2}H - E_1$. We have $w^2 = -2$, so w or $-w$ is effective. Note that $w \cdot H = (-w) \cdot H = 0$. Because the projection $X_I \to \mathbb{P}^5$ is an isomorphism onto a complete intersection of three quadrics, H is strictly ample on X_I. Its intersection with any effective curve must be strictly positive, providing an immediate contradiction.

Thus, no such fractional class exists, and $S_{\mathbb{Q}(i)}(2)$ is primitive. □

Proposition 5.3. *Let X_{II} be a type II surface, and set*

$$G_i := g_i|_{X_{II}} \quad (i = 1, 2, 3, 4).$$

The lattice $G_{II} = \langle G_1, G_2, G_3, G_4 \rangle$ generated by these restrictions has the Gram matrix

$$G_{II} = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}.$$

Moreover, $G_{II} \cong S_{\mathbb{Q}(\sqrt{-3})}(2)$ and $S_{\mathbb{Q}(\sqrt{-3})}(2)$ is primitive in $\mathrm{Pic}(X_{II})$.

Proof. In the Chow ring of $(\mathbb{P}^1)^4$, $g_i^2 = 0$ and the class of X_{II} is $[X_{II}] = (g_1+g_2+g_3+g_4)^2$. Consequently, $G_i^2 = 0$. For $i \neq j$, $G_i \cdot G_j = [g_1g_2g_3g_4]\, g_i g_j (g_1+g_2+g_3+g_4)^2 = 2$. With respect to the unimodular change of basis G_1, G_2, $G_1 + G_2 - G_3$, $-G_1 - G_2 + G_4$, the Gram matrix becomes $U(2) \perp A_2(-2)$.

To show primitivity, assume $S_{\mathbb{Q}(\sqrt{-3})}(2)$ is not primitive. There must exist $v \in S_{\mathbb{Q}(\sqrt{-3})}^\vee(2) \setminus S_{\mathbb{Q}(\sqrt{-3})}(2)$ such that $v \in \mathrm{Pic}(X_{II})$ and $v^2 \in 2\mathbb{Z}$. A direct computation gives $v \equiv \frac{1}{2}(G_1+G_2+G_3+G_4) \pmod{S_{\mathbb{Q}(\sqrt{-3})}(2)}$. Set

$$w := \frac{1}{2}G_1 + \frac{1}{2}G_2 - \frac{1}{2}G_3 - \frac{1}{2}G_4.$$

Then $w \in \mathrm{Pic}(X_{II})$ and $w^2 = -2$. By the Riemann-Roch theorem for $K3$ surfaces, either w or $-w$ is an effective class. However, $w \cdot G_1 = -1$ and $(-w) \cdot G_3 = -1$. Because each G_i is a pullback of a globally generated line bundle from the factors, they are nef and must intersect any effective class non-negatively. This geometric contradiction proves $S_{\mathbb{Q}(\sqrt{-3})}(2)$ is primitive. □

Next, we show that for very general members of these families, the Picard group is generated by the pullbacks of the hyperplane sections.

Theorem 5.4. *For both the Type I and Type II families, the global period map is generically dominant onto the corresponding lattice-polarized $K3$ moduli space. Consequently, a very general Type I surface has Picard lattice $S_{\mathbb{Q}(i)}(2)$, and a very general Type II surface has Picard lattice $S_{\mathbb{Q}(\sqrt{-3})}(2)$.*

Proof. Let G_I and G_{II} be the primitive lattices generated by the ambient divisor classes on the Type I and Type II complete intersections. By Propositions 5.2 and 5.3, we have

$$G_I \cong S_{\mathbb{Q}(i)}(2), \qquad G_{II} \cong S_{\mathbb{Q}(\sqrt{-3})}(2),$$

and both lattices have rank 4. Hence the corresponding lattice-polarized $K3$ moduli spaces have dimension $20-4=16$.

We next compute the dimensions of the two parameter spaces modulo ambient automorphisms. For Type I, the equations are given by three pencils

$$\langle D_{i1}, D_{i2}\rangle \subset H^0\big(\mathbb{P}^5\times(\mathbb{P}^1)^3, \mathcal{O}(h+e_i)\big), \qquad i=1,2,3.$$

Each vector space has dimension 12, so the parameter space is $\mathrm{Gr}(2,12)^3$. After quotienting by $\mathrm{PGL}_6\times\mathrm{PGL}_2^3$, the dimension is

$$3\dim\mathrm{Gr}(2,12) - \dim\mathrm{PGL}_6 - 3\dim\mathrm{PGL}_2 = 60-35-9=16.$$

For Type II, the equations form a pencil in $H^0\big((\mathbb{P}^1)^4, \mathcal{O}(1,1,1,1)\big)$, which has dimension 16. Hence the parameter space is $\mathrm{Gr}(2,16)$, and after quotienting by PGL_2^4 its dimension is

$$\dim\mathrm{Gr}(2,16) - 4\dim\mathrm{PGL}_2 = 28-12=16.$$

It remains to show that the period maps have finite generic fibers. We claim that a general surface in either family reconstructs its ambient complete intersection model from the polarized $K3$ surface together with the ambient divisor classes.

For Type II, let $X_{II}\subset\mathbb{P}_{II} := (\mathbb{P}^1)^4$ be a complete intersection of two divisors of multidegree $(1,1,1,1)$. The Koszul resolution of $\mathcal{I}_{X_{II}}$ is

$$0\longrightarrow\mathcal{O}(-2,-2,-2,-2)\longrightarrow\mathcal{O}(-1,-1,-1,-1)^{\oplus 2}\longrightarrow\mathcal{I}_{X_{II}}\longrightarrow 0.$$

Tensoring by any coordinate divisor g_j gives line bundles with an $\mathcal{O}_{\mathbb{P}^1}(-1)$-factor. Therefore, by the Künneth formula, $H^k(\mathbb{P}_{II}, \mathcal{I}_{X_{II}}(g_j)) = 0 \quad \text{for } k=0,1$. Thus

$$H^0(\mathbb{P}_{II}, \mathcal{O}_{\mathbb{P}_{II}}(g_j)) \xrightarrow{\sim} H^0(X_{II}, \mathcal{O}_{X_{II}}(g_j)).$$

Hence the four pencils $|g_1|,\dots,|g_4|$ on X_{II} recover the map $X_{II}\longrightarrow(\mathbb{P}^1)^4$. The image is the original complete intersection.

For Type I, let $X_I\subset\mathbb{P}_I := \mathbb{P}^5\times(\mathbb{P}^1)^3$ be the complete intersection cut out by the rank-six bundle

$$\mathcal{E} = \bigoplus_{i=1}^{3}\mathcal{O}(h+e_i)^{\oplus 2}.$$

The Koszul resolution of $\mathcal{I}_{X_I}$ has terms $\bigwedge^a\mathcal{E}^\vee$. After tensoring by h or by one of the e_i, every summand has either an $\mathcal{O}_{\mathbb{P}^1}(-1)$-factor or a $\mathbb{P}^5$-factor $\mathcal{O}_{\mathbb{P}^5}(-m)$ with $1\le m\le 5$, and hence has no cohomology by the Künneth formula. Therefore

$$H^k(\mathbb{P}_I, \mathcal{I}_{X_I}(D)) = 0 \qquad \text{for } k=0,1$$

and for $D\in\{h,e_1,e_2,e_3\}$. Consequently,

$$H^0(\mathbb{P}_I, \mathcal{O}_{\mathbb{P}_I}(D)) \xrightarrow{\sim} H^0(X_I, \mathcal{O}_{X_I}(D))$$

for $D=h,e_1,e_2,e_3$. Thus the complete linear systems $|h|,\ |e_1|,\ |e_2|,\ |e_3|$ on X_I reconstruct the ambient embedding $X_I\longrightarrow\mathbb{P}^5\times(\mathbb{P}^1)^3$.

Thus a very general fiber of the period map is finite: indeed, once the ordered ambient divisor classes are fixed by the lattice polarization, the above Koszul reconstruction determines the ambient projective model and the defining equations up to finitely many choices; forgetting the ordering can only quotient by a finite group of lattice symmetries.

The source and target of each period map have dimension 16. Since the period maps have finite generic fibers, their images also have dimension 16. Hence the period maps are generically dominant. In particular, the differential of the period map has rank 16 at a general point.

Finally, the locus where the Picard lattice strictly contains the primitive ambient lattice is a countable union of proper Noether–Lefschetz divisors. Therefore a very general member has no algebraic classes beyond the ambient lattice. Hence

$$\mathrm{Pic}(X_I) = G_I \cong S_{\mathbb{Q}(i)}(2), \qquad \mathrm{Pic}(X_{II}) = G_{II} \cong S_{\mathbb{Q}(\sqrt{-3})}(2).$$

□

We now produce involutions from pairs of genus-one pencils.

Lemma 5.5. *Let X be a smooth projective $K3$ surface, and let $F, G \in \mathrm{NS}(X)$ be primitive, nef, isotropic divisor classes such that $F^2 = G^2 = 0$ and $F \cdot G = 2$. Then:*

1. *The complete linear systems $|F|$ and $|G|$ define genus-one fibrations $\varphi_F, \varphi_G \colon X \to \mathbb{P}^1$.*
2. *The product morphism $\Phi = (\varphi_F, \varphi_G) \colon X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is generically finite of degree 2.*
3. *The nontrivial deck transformation of Φ defines an automorphism of X of order two.*

Proof. By [6, Proposition 3.10] a primitive nef divisor F on a $K3$ surface with $F^2 = 0$ defines a base-point-free genus-one pencil. These define the fibrations φ_F and φ_G. Let H_1, H_2 be the ruling classes on $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\Phi^* H_1 = F$ and $\Phi^* H_2 = G$, which yields $\Phi^*(H_1 + H_2) = F + G$. Since $(F+G)^2 = F^2 + 2F\cdot G + G^2 = 4 > 0$, the divisor $F+G$ is big, meaning Φ is generically finite. By the projection formula, $(F+G)^2 = \deg(\Phi)(H_1+H_2)^2$, which implies $4 = 2\deg(\Phi)$, so $\deg(\Phi) = 2$.

The degree-two function field extension $\mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{C}(X)$ induces a nontrivial birational deck involution on X. Because X is a smooth $K3$ surface, every birational self-map is a regular automorphism. □

Proposition 5.6. *A very general type I surface admits 12 natural deck involutions arising from pairs of isotropic classes with intersection 2. A very general type II surface admits 6 such natural deck involutions.*

Furthermore, each distinguished isotropic class corresponds to an ideal vertex of the associated regular ideal octahedron, respectively regular ideal tetrahedron.

Proof. For type I, Proposition 5.2 gives six distinguished primitive isotropic classes $E_1, E_2, E_3, H - E_1, H - E_2, H - E_3$. They lie on the boundary of the positive cone, since each has square zero. Equivalently, after projectivizing the positive cone, their rays determine ideal boundary points of the associated hyperbolic space $\mathbb{H}(S_I) = \{x \in S_I \otimes \mathbb{R} \mid x^2 > 0\}/\mathbb{R}_{>0}$. The intersection table from Proposition 5.2 shows that, among these six classes, the three pairs $\{E_i, H - E_i\}, i = 1, 2, 3$, are the only non-adjacent pairs, while every other pair has intersection number 2. Therefore the graph whose vertices are the six isotropic rays and whose edges are the pairs with intersection 2 is the complete graph on six vertices with three disjoint edges removed. This is precisely the one-skeleton of an octahedron, with opposite vertices given by E_i and $H - E_i$. Thus the six natural isotropic rays are naturally identified with the six ideal vertices of an ideal octahedron. Since all edge-pairs have the same intersection number, this is the regular ideal octahedron in the hyperbolic model determined by S_I. Hence the 12 pairs with intersection 2 are exactly the 12 edges of this ideal regular octahedron.

For type II, the four ambient classes G_1, G_2, G_3, G_4 satisfy $G_i^2 = 0, \qquad G_i \cdot G_j = 2 \quad (i \neq j)$. Thus their projective rays are four ideal boundary points of $\mathbb{H}(S_{II})$. Since every pair of distinct vertices has the same intersection number, the complete graph on these four isotropic rays is the one-skeleton of a regular ideal tetrahedron. Consequently the classes G_1, G_2, G_3, G_4 are naturally identified with the four ideal vertices of this tetrahedron, and the six pairs $\{G_i, G_j\}$ with $i \neq j$ are exactly its six edges.

Finally, by Lemma 5.5, every adjacent pair of primitive nef isotropic classes F, G with $F \cdot G = 2$ gives a degree-two morphism $X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and hence a deck involution. Therefore the type I surface has one natural deck involution for each edge of the ideal octahedron, namely 12, and the type II surface has one natural deck involution for each edge of the ideal tetrahedron, namely 6. □

We now bridge the deck involution with a purely algebraic edge half-turn on the lattice.

Lemma 5.7. *Let X be a smooth K3 surface with Picard rank $\rho(X) = 4$, and let $f\colon X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a finite degree-two cover with deck involution ι. Let $\alpha = f^*H_1$ and $\beta = f^*H_2$ be the pullbacks of the two ruling classes of $\mathbb{P}^1 \times \mathbb{P}^1$. Then α and β are primitive isotropic classes satisfying $(\alpha, \beta) = 2$, and the induced action of ι^* on $\mathrm{NS}(X)$ is an involutive isometry given by the formula:*

$$\iota^*(x) = -x + (x, \beta)\alpha + (x, \alpha)\beta$$

Proof. First, we determine the ι-invariant subspace of the Néron–Severi group. For a finite morphism f of degree two, the pullback and pushforward operators satisfy $f_*f^* = 2\,\mathrm{id}$ on $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$ and $f^*f_* = \mathrm{id} + \iota^*$ on $H^2(X, \mathbb{Q})$. If a class $x \in H^2(X, \mathbb{Q})$ is ι-invariant ($x = \iota^*x$), then:

$$x = \frac{1}{2}(x + \iota^*x) = \frac{1}{2}f^*f_*x$$

Thus, every ι-invariant cohomology class is pulled back from $\mathbb{P}^1 \times \mathbb{P}^1$. Since the reverse inclusion is immediate (as ι acts trivially on the quotient), and $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q}) = \mathbb{Q}H_1 \oplus \mathbb{Q}H_2$ where both classes are algebraic, we obtain:

$$\mathrm{NS}(X)^{\iota}_{\mathbb{Q}} = f^*\,\mathrm{NS}(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{Q}} = \mathbb{Q}\alpha \oplus \mathbb{Q}\beta$$

Next, consider the purely algebraic map defined by $\phi(x) := -x + (x, \beta)\alpha + (x, \alpha)\beta$. Direct calculation shows that $\phi(\alpha) = \alpha$ and $\phi(\beta) = \beta$. For any x in the orthogonal complement $(\mathbb{Q}\alpha \oplus \mathbb{Q}\beta)^{\perp}$, we have $(x, \alpha) = (x, \beta) = 0$, meaning $\phi(x) = -x$. Since this gives an orthogonal direct-sum decomposition and ϕ preserves the intersection form on each summand, ϕ is an involutive isometry.

Finally, we characterize the geometric involution ι^* on $\mathrm{NS}(X)_{\mathbb{Q}}$. Because $\rho(X) = 4$, the orthogonal complement $K := (\mathbb{Q}\alpha \oplus \mathbb{Q}\beta)^{\perp}$ has dimension 2. The involution ι^* preserves K. Since we established that the invariant subspace $\mathrm{NS}(X)^{\iota}_{\mathbb{Q}}$ is exactly $\mathbb{Q}\alpha \oplus \mathbb{Q}\beta$, ι^* possesses no non-zero invariant vectors in K, so that it must act as $-\mathrm{id}_K$. Therefore, ι^* acts as the identity on $\mathbb{Q}\alpha \oplus \mathbb{Q}\beta$ and as minus the identity on K, which agrees with the behavior of the algebraic map ϕ. We conclude that $\iota^*(x) = -x + (x, \beta)\alpha + (x, \alpha)\beta$. Under the hyperbolic realization of the positive cone, this involutive isometry corresponds exactly to the hyperbolic half-turn about the geodesic joining the two ideal points $[\alpha]$ and $[\beta]$. □

We now show that, in each case, these involutions generate the automorphism group $\mathrm{Bi}_K(2)$. We do this by comparing arithmetic indices and hyperbolic covolumes. By Lemma 5.7, each natural deck involution associated with a pair of ambient isotropic classes is the corresponding edge half-turn in the hyperbolic model of the positive cone.

5.1. $K = \mathbb{Q}(i)$. Let $O \subset \mathbb{H}^3$ be a regular ideal octahedron. For each face F of O, let s_F denote the hyperbolic reflection in the totally geodesic plane containing F. Define $W_O := \langle s_F \mid F \text{ is a face of } O\rangle$. The reflections in the faces of a regular ideal octahedron generate the Coxeter reflection group of the regular ideal octahedral tessellation of $\mathbb{H}^3$.

Lemma 5.8. *Let F_1 and F_2 be two faces of O meeting along an edge e. Then $s_{F_1}s_{F_2} = \iota_e$, where ι_e is the edge half-turn about e.*

Proof. A regular ideal octahedron has dihedral angles of $\pi/2$. In hyperbolic space, the product of two reflections in totally geodesic planes meeting at angle θ is the elliptic rotation of angle 2θ about their intersection. With $\theta = \pi/2$, $s_{F_1}s_{F_2}$ is the rotation of angle π about e, which is precisely ι_e. □

The involutions $\iota_{\alpha,\beta}$ from the previous section are precisely the edge half-turns of O. This description is encoded by the sign homomorphism $\varepsilon_O : W_O \to \{\pm 1\}$ defined by $\varepsilon_O(s_F) = -1$. We define the even Coxeter subgroup $W_O^+ := \ker(\varepsilon_O)$. Thus W_O^+ consists of elements expressible as products of an even number of face reflections.

Lemma 5.9. *The even Coxeter subgroup W_O^+ is generated by the edge half-turns of the regular ideal octahedron O.*

Proof. By definition, W_O^+ is generated by products of two face reflections. Since the face-adjacency graph of the octahedron is connected, it suffices to take products of reflections in adjacent faces. By Lemma 5.8, the product of reflections in two adjacent faces is precisely the half-turn about their common edge. □

Proposition 5.10. *Choose a face F_0 of O, and let s_{F_0} be the reflection in the plane containing F_0. The set $D := O \cup s_{F_0}(O)$ is a fundamental domain for the action of W_O^+ on $\mathbb{H}^3$.*

Proof. Since W_O^+ has index two in W_O with coset decomposition $W_O = W_O^+ \sqcup W_O^+ s_{F_0}$, we can regroup the tiling as

$$\mathbb{H}^3 = \bigcup_{g \in W_O^+} g\,(O \cup s_{F_0}(O)) = \bigcup_{g \in W_O^+} g(D).$$

To see that distinct translates have disjoint interiors, suppose $\mathrm{Int}(gD) \cap \mathrm{Int}(hD) \neq \varnothing$ for some $g, h \in W_O^+$. This implies $\mathrm{Int}(g\delta O) \cap \mathrm{Int}(h\delta' O) \neq \varnothing$ for some $\delta, \delta' \in \{1, s_{F_0}\}$. Since O is a fundamental domain for W_O, we must have $g\delta = h\delta'$, or $h^{-1}g = \delta'\delta^{-1}$. The left-hand side lies in W_O^+, but $s_{F_0} \notin W_O^+$. Thus $\delta'\delta^{-1}$ cannot be s_{F_0}, forcing $\delta'\delta^{-1} = 1$ and therefore $g = h$. □

Proposition 5.11. *Let $G_{\mathrm{deck}} \subseteq \mathrm{Bi}_{\mathbb{Q}(i)}(2)$ denote the subgroup generated by the natural deck involutions. Then $G_{\mathrm{deck}} = \mathrm{Bi}_{\mathbb{Q}(i)}(2)$.*

Proof. By Lemma 5.9, we identify G_{deck} with W_O^+. We will prove equality by comparing the arithmetic index $[\mathrm{Bi}_{\mathbb{Q}(i)} : \mathrm{Bi}_{\mathbb{Q}(i)}(2)]$ with the geometric index $[\mathrm{Bi}_{\mathbb{Q}(i)} : G_{\mathrm{deck}}]$ computed from the octahedral covolume.

Let $R_i := \mathbb{Z}[i]/(2)$. Since $(2) = -(1+i)^2$, we have $R_i \cong \mathbb{F}_2[\epsilon]/(\epsilon^2), \quad \epsilon = 1+i$. The reduction map $\mathrm{GL}_2(\mathbb{Z}[i]) \longrightarrow \mathrm{GL}_2(R_i)$ is surjective. Hence $\mathrm{Bi}_{\mathbb{Q}(i)} / \mathrm{Bi}_{\mathbb{Q}(i)}(2) \cong \mathrm{PGL}_2(R_i)$. The quotient map $R_i \to \mathbb{F}_2$ induces a homomorphism

$$\mathrm{PGL}_2(R_i) \longrightarrow \mathrm{PGL}_2(\mathbb{F}_2) \cong \mathrm{GL}_2(\mathbb{F}_2) \cong S_3.$$

Its kernel consists of projective classes of matrices of the form $I + \epsilon A, \quad A \in M_2(\mathbb{F}_2)$, where two such matrices define the same projective class if they differ by a scalar matrix $1 + \epsilon\lambda, \quad \lambda \in \mathbb{F}_2$. Thus the kernel is $M_2(\mathbb{F}_2)/\mathbb{F}_2 I$, which has order 8. Since $|\mathrm{PGL}_2(\mathbb{F}_2)| = |S_3| = 6$, we obtain $|\mathrm{PGL}_2(R_i)| = 8 \cdot 6 = 48$. Therefore $[\mathrm{Bi}_{\mathbb{Q}(i)} : \mathrm{Bi}_{\mathbb{Q}(i)}(2)] = 48$.

We now compare this with the geometric index. By Proposition 5.10, the group $G_{\mathrm{deck}} = W_O^+$ has fundamental polyhedron $D = O \cup s_{F_0}(O)$, where O is the regular ideal octahedron. Hence $\mathrm{Vol}(G_{\mathrm{deck}}\backslash\mathbb{H}^3) = \mathrm{Vol}(D) = 2\,\mathrm{Vol}(O)$. By [8, Section 4.5]

$$\mathrm{Vol}(\mathrm{Bi}_{\mathbb{Q}(i)}\backslash\mathbb{H}^3) = \frac{1}{24}\,\mathrm{Vol}(O).$$

Therefore

$$[\mathrm{Bi}_{\mathbb{Q}(i)} : G_{\mathrm{deck}}] = \frac{\mathrm{Vol}(G_{\mathrm{deck}}\backslash\mathbb{H}^3)}{\mathrm{Vol}(\mathrm{Bi}_{\mathbb{Q}(i)}\backslash\mathbb{H}^3)} = \frac{2\,\mathrm{Vol}(O)}{\mathrm{Vol}(O)/24} = 48.$$

Thus both the arithmetic and geometric indices are 48. Since we already have the inclusion $G_{\mathrm{deck}} \subseteq \mathrm{Bi}_{\mathbb{Q}(i)}(2)$, and both subgroups have index 48 in $\mathrm{Bi}_{\mathbb{Q}(i)}$, it follows that $G_{\mathrm{deck}} = \mathrm{Bi}_{\mathbb{Q}(i)}(2)$. □

5.2. **The case $K = \mathbb{Q}(\sqrt{-3})$.** Let $\omega = \frac{-1+\sqrt{-3}}{2}$, $\mathcal{O}_K = \mathbb{Z}[\omega]$. We use the regular ideal cubic honeycomb in $\mathbb{H}^3$. A regular ideal cube has dihedral angle $\pi/3$, so six cubes meet around each edge; in Schläfli notation this is the honeycomb $\{4,3,6\}$. The reflections in the square faces of one regular ideal cube generate the corresponding Coxeter reflection group. We use Coxeter's standard description of this honeycomb [4]. In the Eisenstein normalization, take the regular ideal tetrahedron

$$T = [\infty, 0, 1, -\omega].$$

The regular ideal cube can be obtained from T by adjoining the four tetrahedra adjacent to T across its four faces. Namely, if s_F denotes reflection in the plane containing a face $F \subset T$, then

$$D = T \cup \bigcup_{F \subset T} s_F(T)$$

is a regular ideal cube, subdivided into five regular ideal tetrahedra.

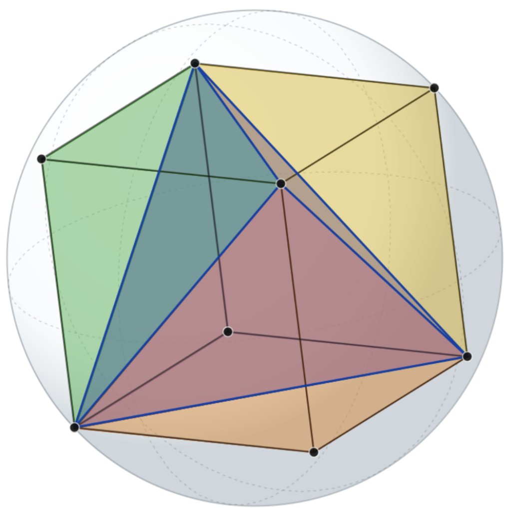

FIGURE 2. The union of the central ideal tetrahedron T with its four adjacent Coxeter chambers $T_1, \ldots, T_4$.

Lemma 5.12. *Let $T \subset \mathbb{H}^3$ be a regular ideal tetrahedron, and let*

$$D := T \cup \bigcup_{F \subset T} s_F(T),$$

where s_F denotes reflection in the plane containing the face F of T. Then D is a regular ideal cube. Moreover, if e runs over the six edges of T, the half-turn ι_e about e gives the side pairing of the square face of D having e as a diagonal. Consequently, the group

$$G_T := \langle \iota_e \mid e \subset T \text{ an edge} \rangle$$

has D as a fundamental polyhedron.

Proof. By Coxeter's description of the regular ideal cubic honeycomb $\{4,3,6\}$, a regular ideal cube is subdivided into five regular ideal tetrahedra: one central tetrahedron T, together with the four tetrahedra adjacent to T across its four faces [4]. Hence

$$D = T \cup \bigcup_{F \subset T} s_F(T)$$

is a regular ideal cube.

The boundary of D consists of twelve triangular faces, grouped into six squares. These six square faces are naturally indexed by the six edges of the central tetrahedron T. If $e \subset T$ is an edge, let Q_e be the corresponding square face of D. Then e is a diagonal of Q_e. The half-turn ι_e about e preserves the plane containing Q_e and exchanges the two half-spaces bounded by this plane. Therefore ι_e preserves Q_e setwise and exchanges the two half-spaces bounded by the plane containing Q_e. Hence $\iota_e(D)$ is the cube adjacent to D across Q_e.

Thus the six half-turns ι_e, one for each edge $e \subset T$, are exactly the side-pairing transformations of the six square faces of the cube D. Since the regular ideal cubic honeycomb $\{4,3,6\}$ has dihedral angle $\pi/3$, six cubes meet around each edge. Hence the side pairings satisfy the cycle conditions of the Poincaré polyhedron theorem [9, cf. Theorem H.11]. Therefore the group $G_T = \langle \iota_e \mid e \subset T \rangle$ is discrete and has D as a fundamental polyhedron.

□

Proposition 5.13. *Let $G_{\text{deck}} \subseteq \text{Bi}_{\mathbb{Q}(\sqrt{-3})}(2)$ denote the subgroup generated by the natural deck involutions. Then $G_{\text{deck}} = \text{Bi}_{\mathbb{Q}(\sqrt{-3})}(2)$.*

Proof. By Corollary 3.16, $G_{\text{deck}} \subset \text{Bi}_{\mathbb{Q}(\sqrt{-3})}$. By Lemma 5.12, the deck involution group G_{deck} is precisely G_T. We verify the subgroup equality by confirming that $[\text{Bi}_{\mathbb{Q}(\sqrt{-3})} : \text{Bi}_{\mathbb{Q}(\sqrt{-3})}(2)]$ coincides with the geometric index $[\text{Bi}_{\mathbb{Q}(\sqrt{-3})} : G_{\text{deck}}]$.

First, we compute the arithmetic index of $\text{Bi}_{\mathbb{Q}(\sqrt{-3})}(2)$. Since 2 is inert in $\mathbb{Z}[\omega]$, we have $\mathbb{Z}[\omega]/(2) \cong \mathbb{F}_4$. The reduction map $\text{PGL}_2(\mathbb{Z}[\omega]) \longrightarrow \text{PGL}_2(\mathbb{F}_4)$ is surjective, so $[\text{Bi}_{\mathbb{Q}(\sqrt{-3})} : \text{Bi}_{\mathbb{Q}(\sqrt{-3})}(2)] = |\text{PGL}_2(\mathbb{F}_4)|$. We calculate $|\operatorname{GL}_2(\mathbb{F}_4)| = (4^2-1)(4^2-4) = 15 \cdot 12 = 180$. Projectivizing by the scalar subgroup of order 3 gives $|\text{PGL}_2(\mathbb{F}_4)| = 180/3 = 60$. Thus, $[\text{Bi}_{\mathbb{Q}(\sqrt{-3})} : \text{Bi}_{\mathbb{Q}(\sqrt{-3})}(2)] = 60$.

On the other hand, we compute the geometric index. By Lemma 5.12, D is a fundamental domain for G_{deck}. Since D is the union of five regular ideal tetrahedra, $\text{Vol}(G_{\text{deck}} \backslash \mathbb{H}^3) = 5v_3$. Using the covolume formula in PGL_2 (see [8, Chapter 1.4.3]), we have $\text{Vol}(\text{Bi}_{\mathbb{Q}(\sqrt{-3})} \backslash \mathbb{H}^3) = v_3/12$. We obtain the index:

$$[\text{Bi}_{\mathbb{Q}(\sqrt{-3})} : G_{\text{deck}}] = \frac{5v_3}{v_3/12} = 60.$$

Since both indices equal 60, the inclusion $G_{\text{deck}} \subseteq \text{Bi}_{\mathbb{Q}(\sqrt{-3})}(2)$ is an exact equality, yielding $G_{\text{deck}} = \text{Bi}_{\mathbb{Q}(\sqrt{-3})}(2)$. □

Theorem 5.14. *For $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-3})$, the group of natural deck involutions G_{deck} generates the full automorphism group $\text{Aut}(X_{K,2})$.*

Proof. By Proposition 5.11 and Proposition 5.13, the geometric subgroup generated by the deck involutions is precisely $G_{\text{deck}} = \text{Bi}_K(2)$ in both cases. By the arithmetic Torelli identification established in Section 3, we have $\text{Aut}(X_{K,2}) \cong \text{Bi}_K(2)$. Therefore, we conclude that $G_{\text{deck}} = \text{Aut}(X_{K,2})$. □

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